Acoustic wave propagation in thin shear layers

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UMR CNRS-ENSTA-INRIA

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Work in collaboration with

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Lauris Joubert, Ricardo Weder

Context and motivation

Aeroacoustics : sound propagation in flows

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⟶ Many applications in aeronautics

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Specific difficulty : modelize the interaction between acoustic waves and walls

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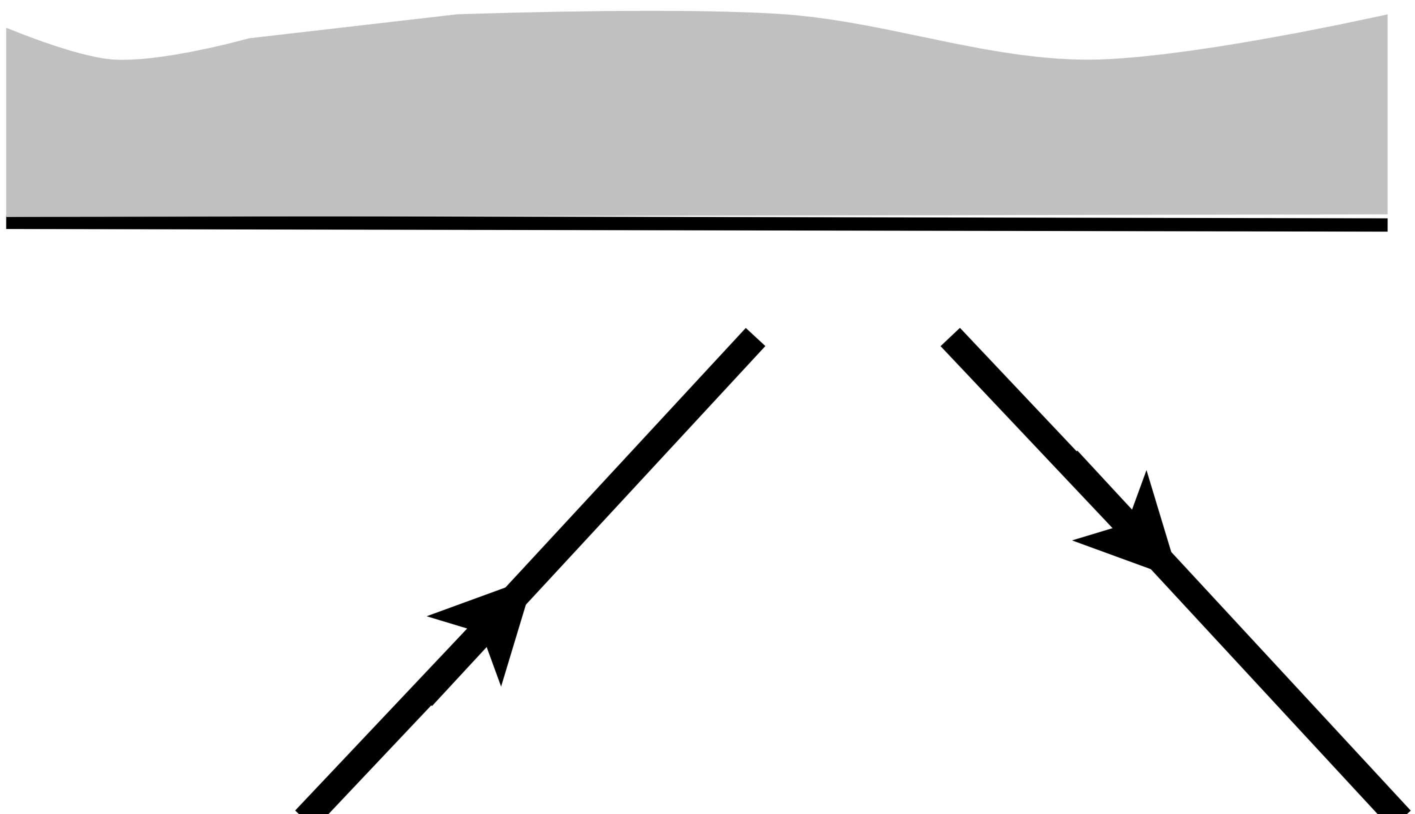

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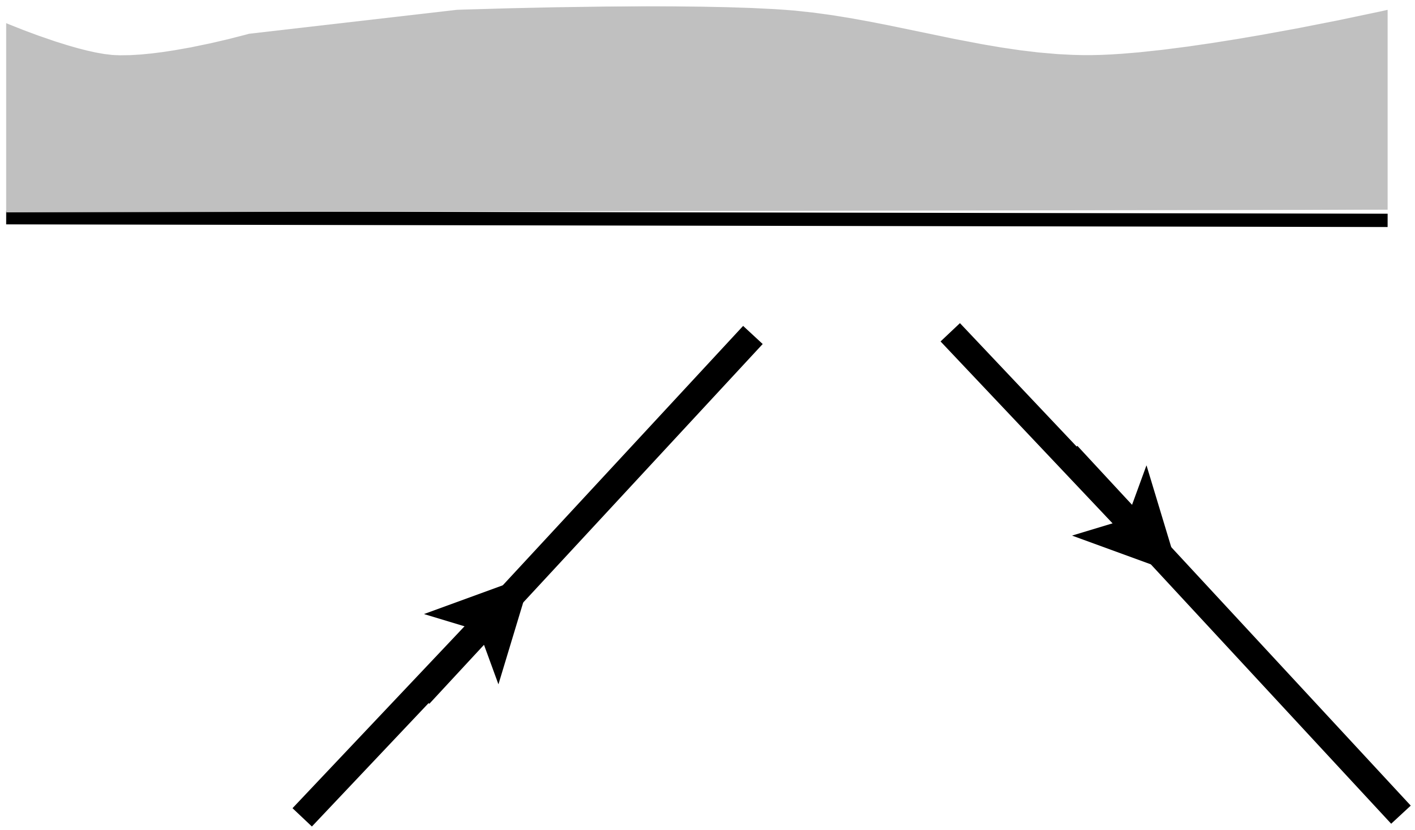

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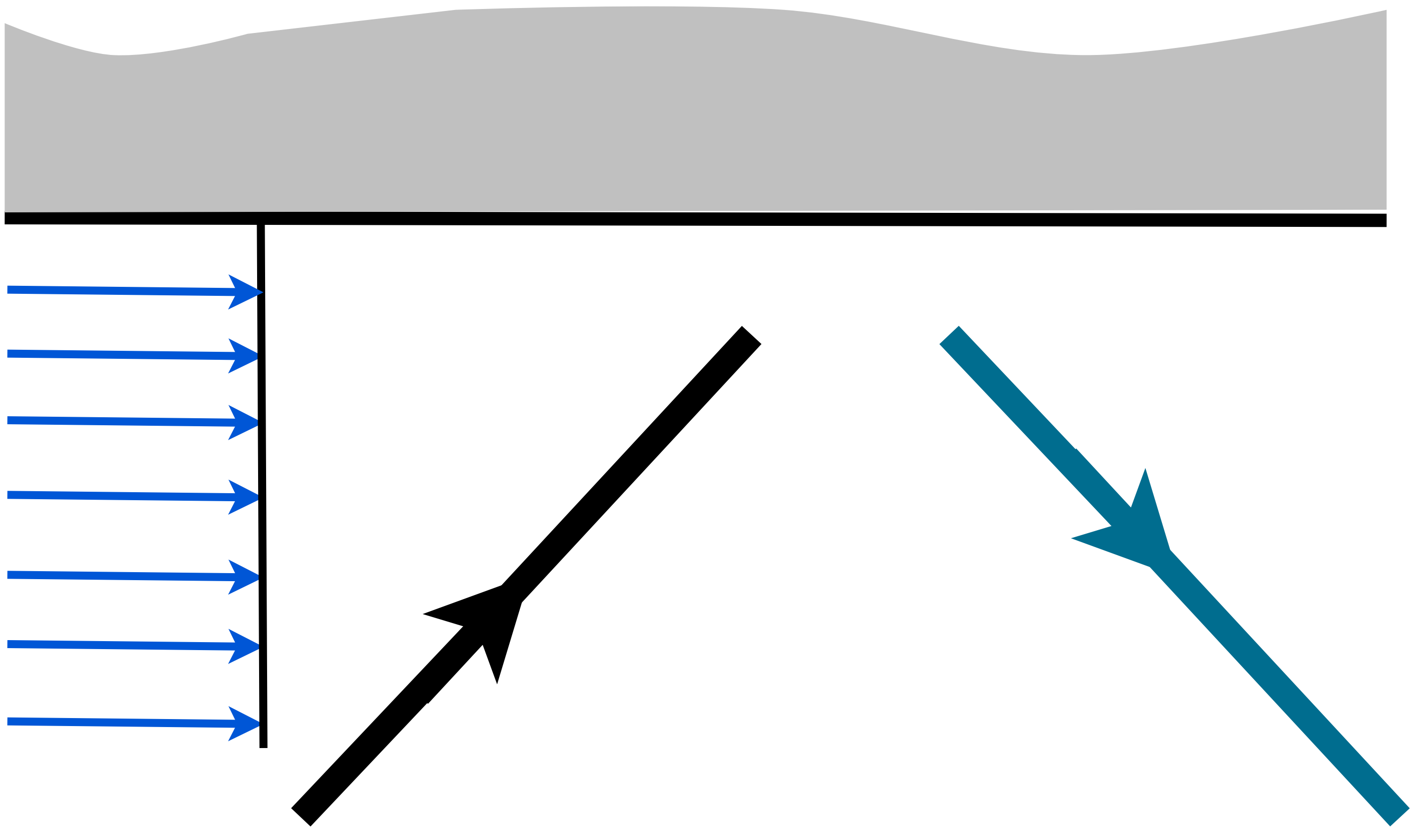

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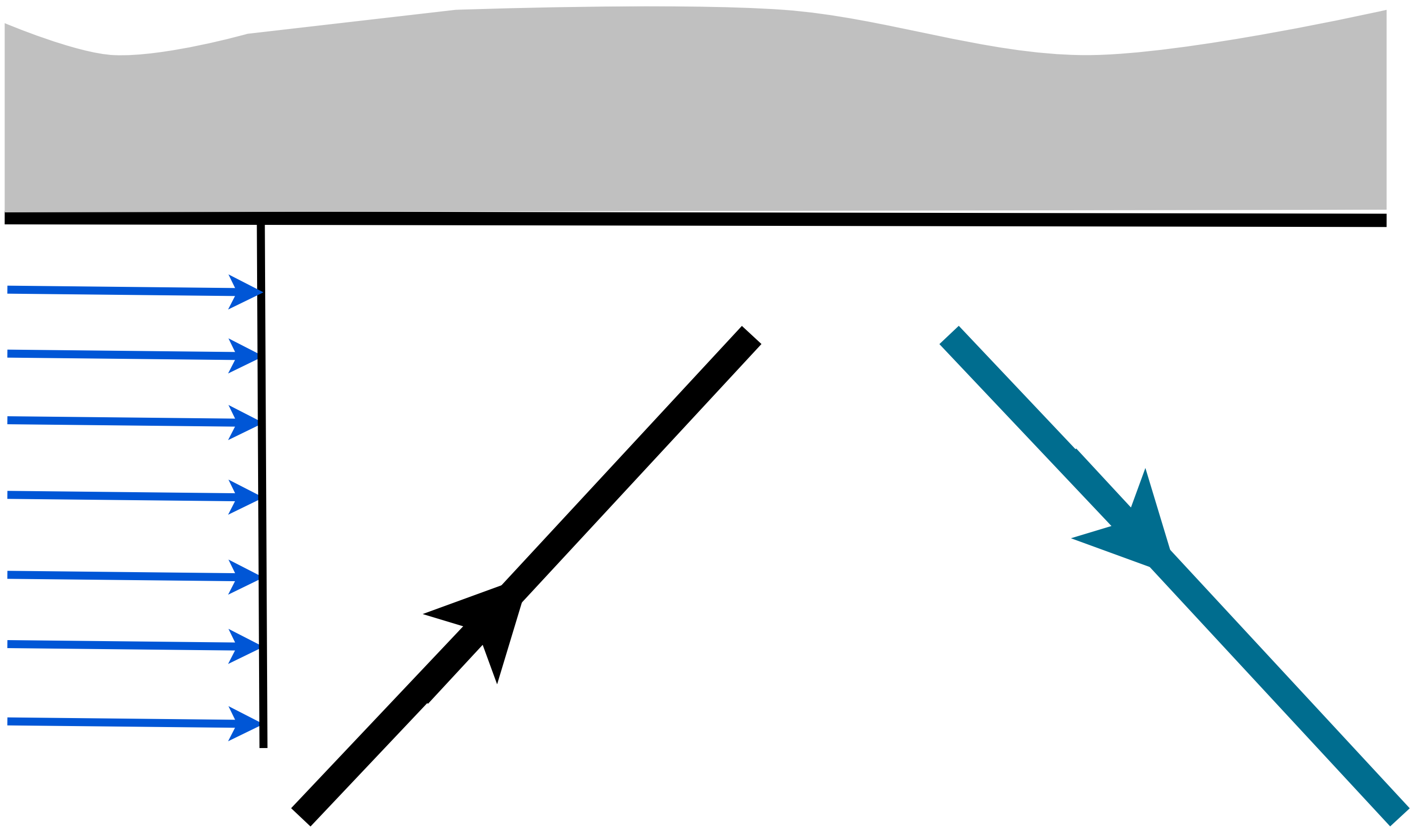

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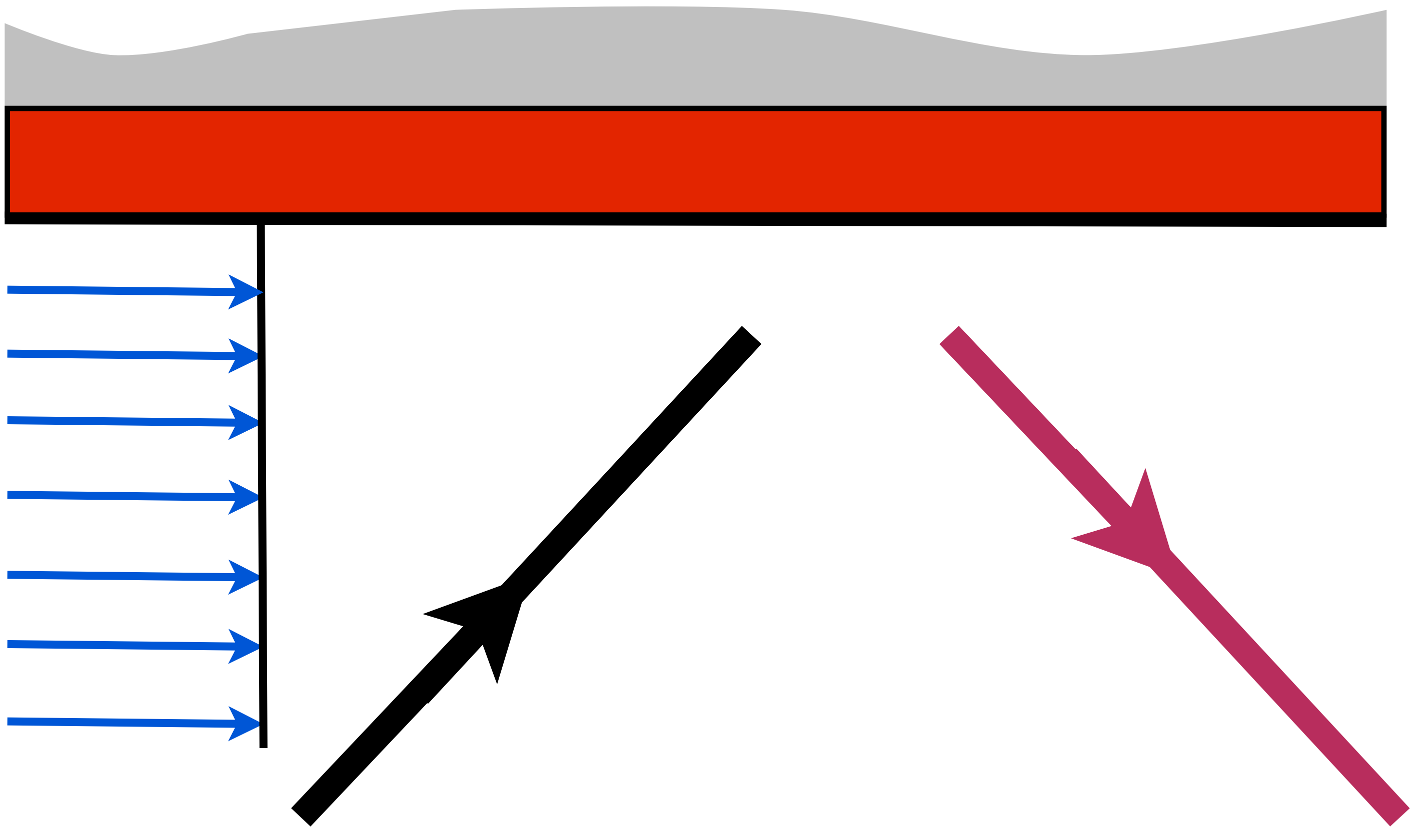

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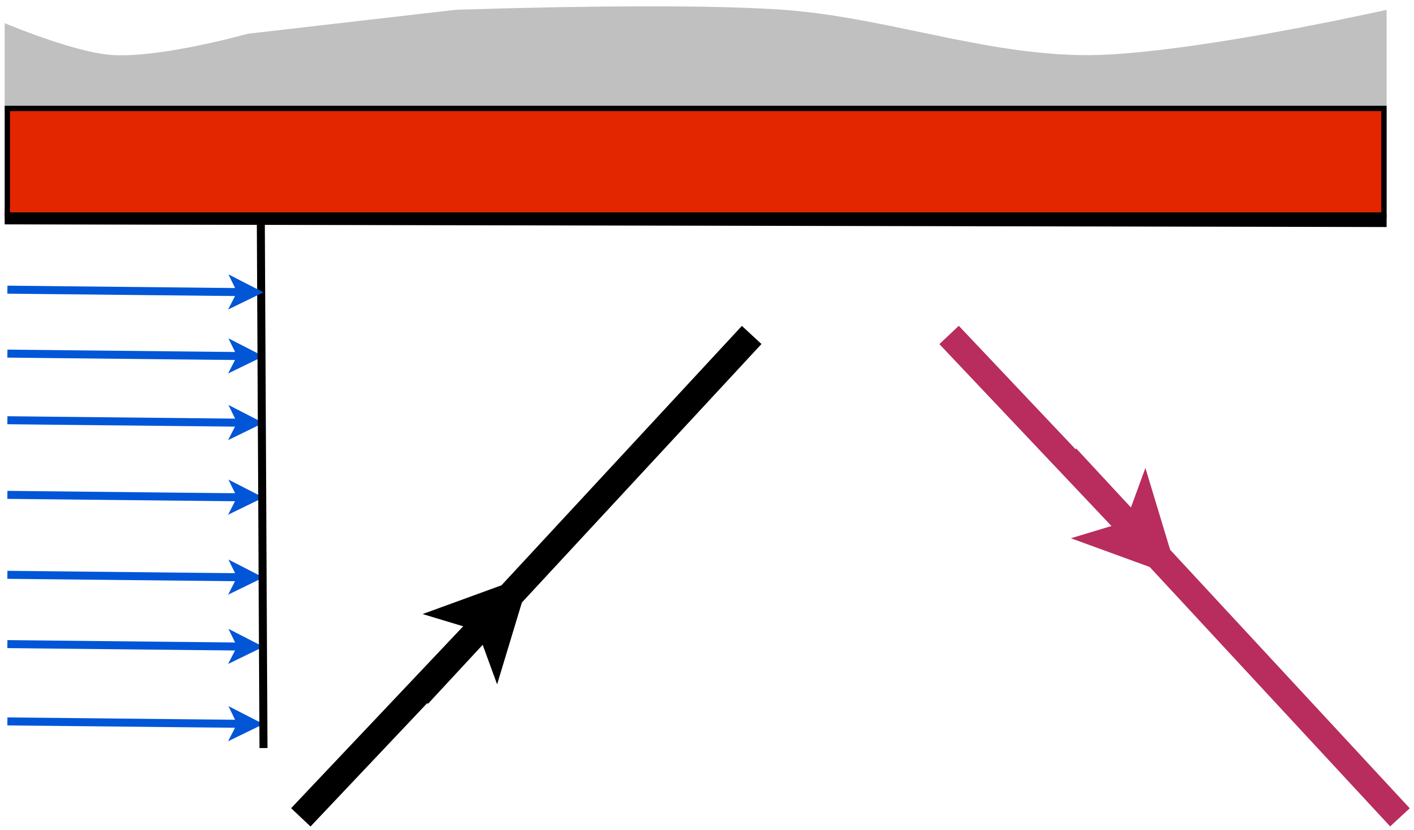

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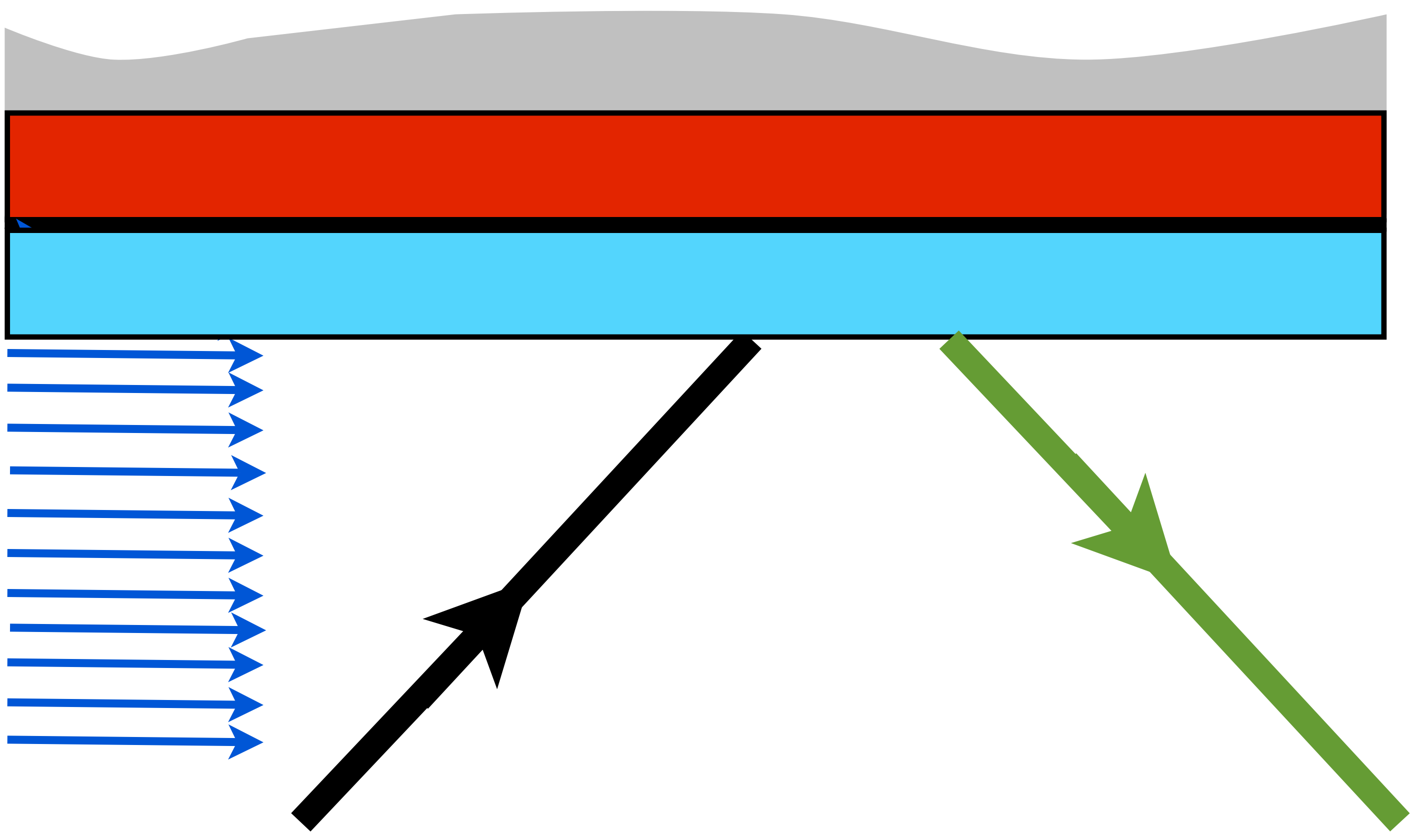

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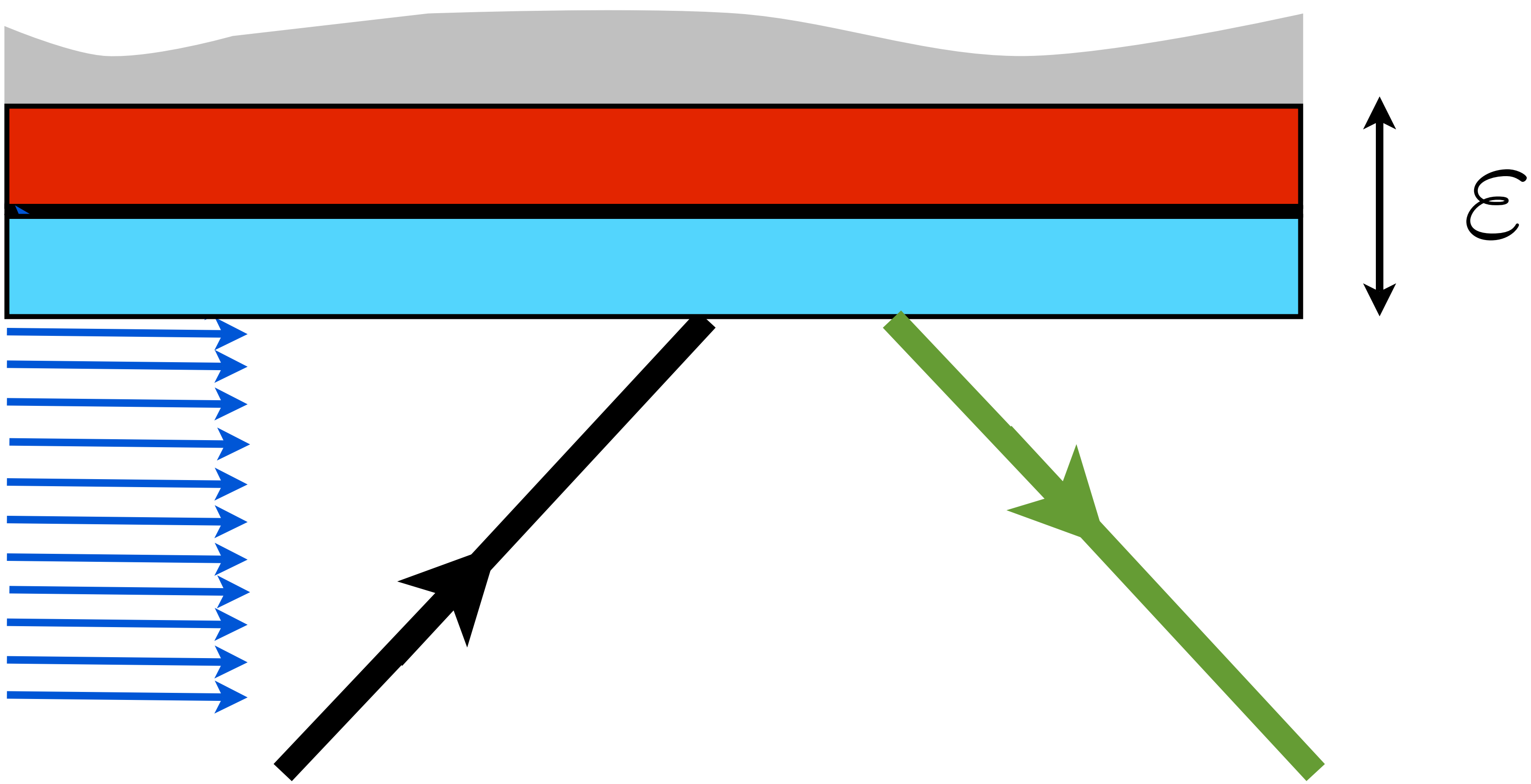

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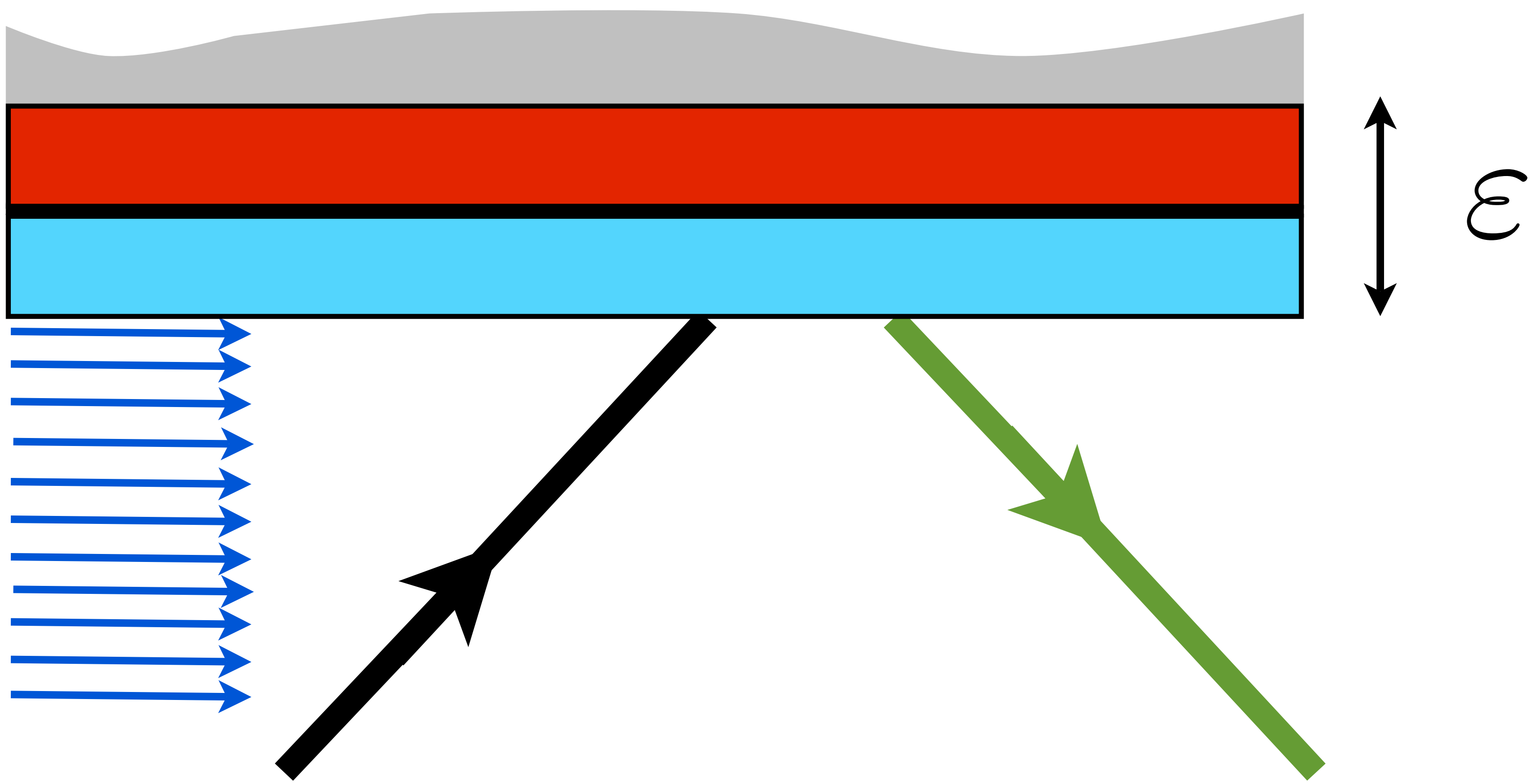

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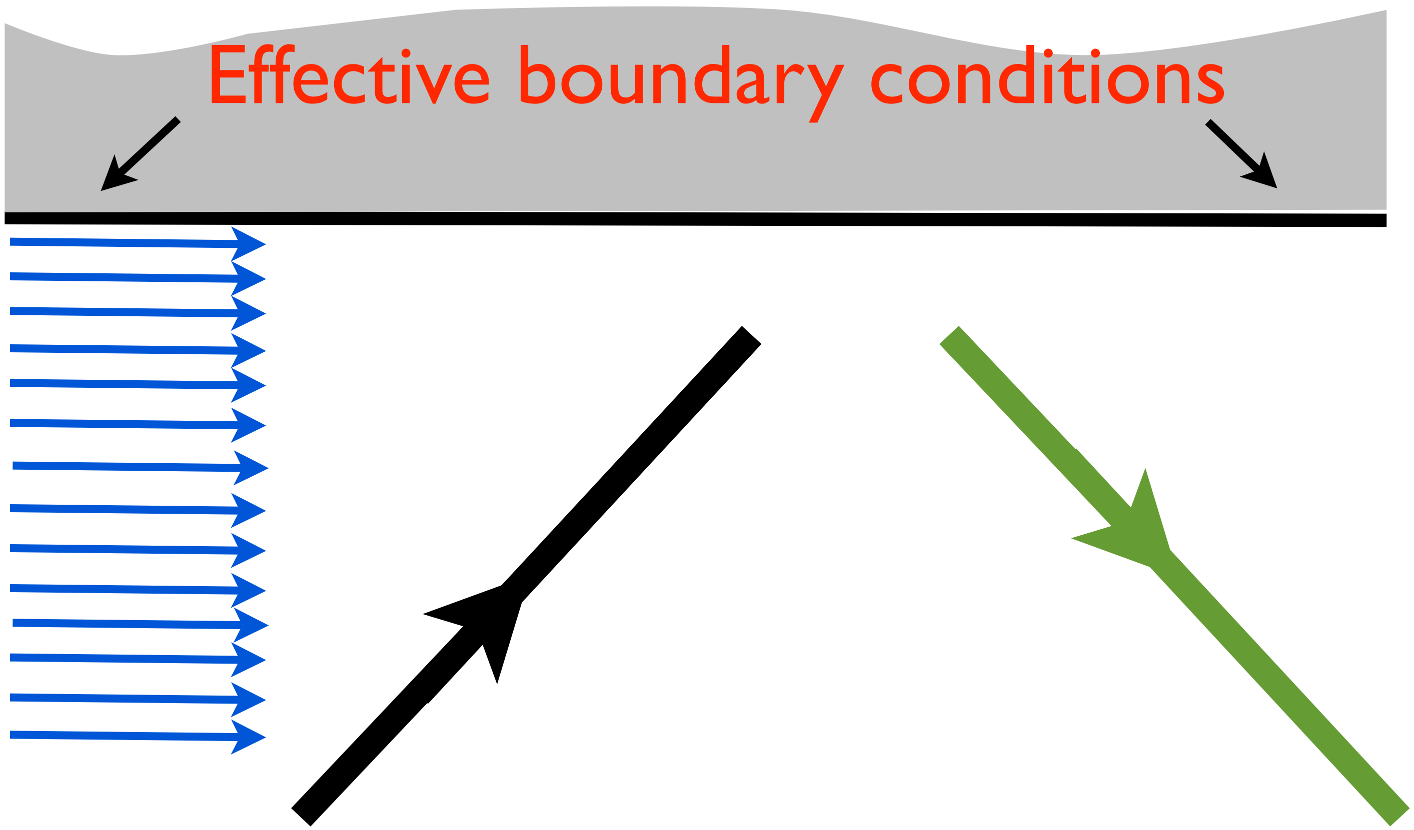

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From E. J. Brambley (J. Sound Vibr. , 2009)

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Objective : derive new lining models using rigorous asymptotic analysis

The mathematical models

Euler
$$\begin{cases} (\partial_t + M \cdot \nabla) v + (v \cdot \nabla) M + \nabla p = 0 \\ (\partial_t + M \cdot \nabla) p + \nabla \cdot v = 0 \end{cases}$$

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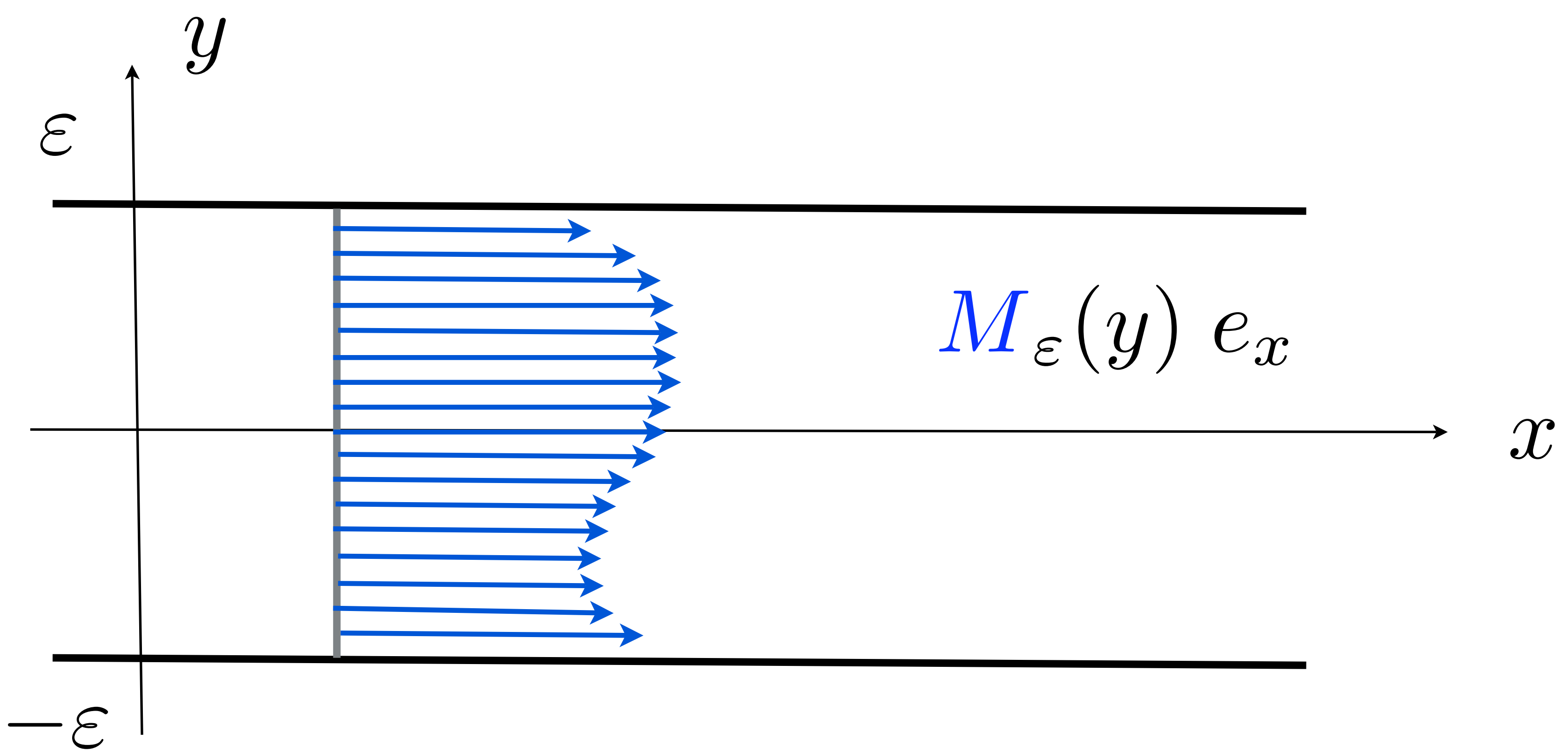
A preliminary analysis : Acoustic wave propagation in a thin duct



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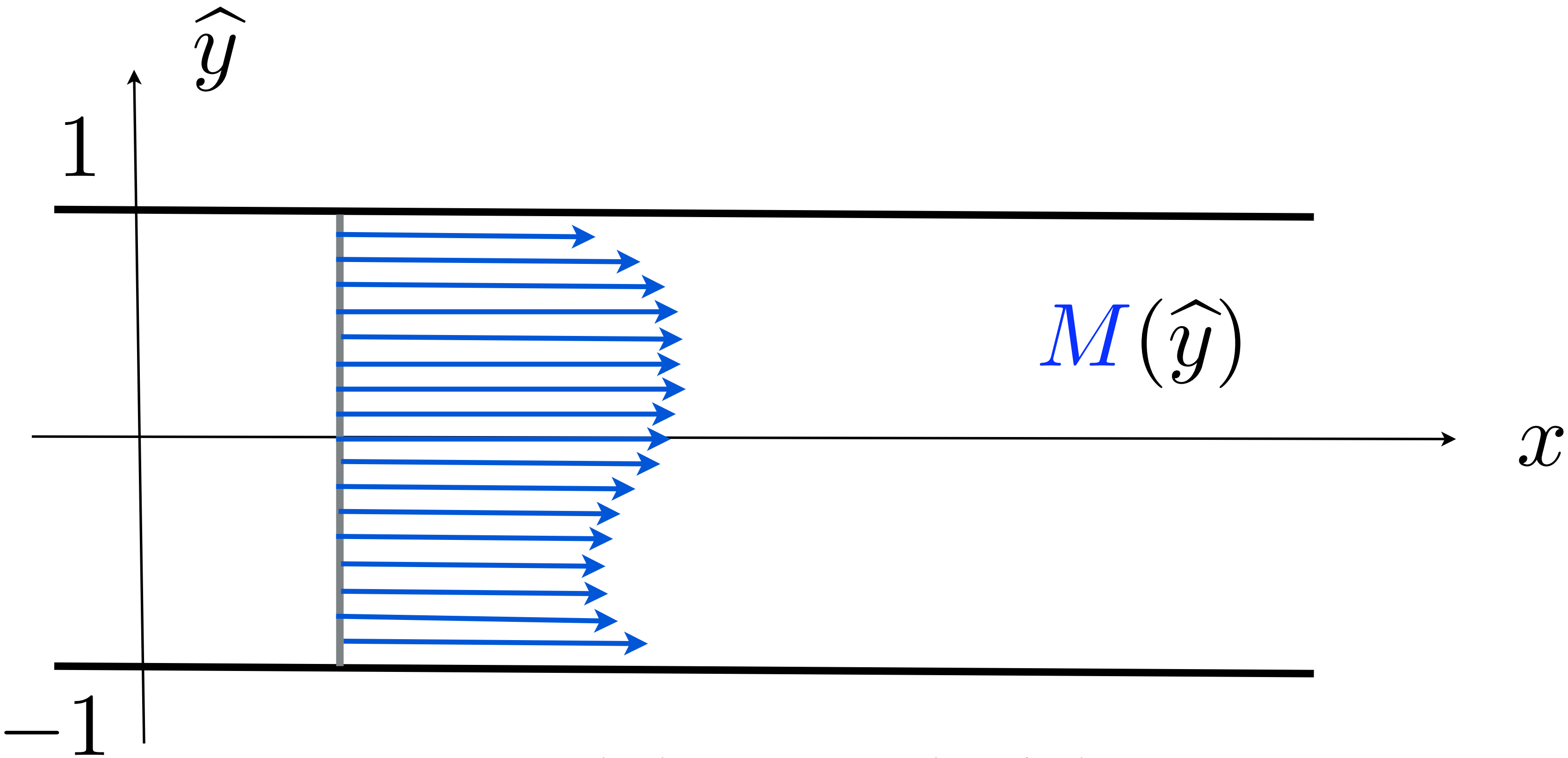

$$M_\varepsilon(y) = M(y/\varepsilon)$$

Galbrun's equations in a 2D thin duct

$$
(\widetilde{\mathcal{P}})_\varepsilon \begin{cases} (\partial_t + M_\varepsilon \partial_x)^2\, \mathbf{u}_\varepsilon - \partial_x(\partial_x \mathbf{u}_\varepsilon + \partial_y \mathbf{v}_\varepsilon) = 0 \\ (\partial_t + M_\varepsilon \partial_x)^2\, \mathbf{v}_\varepsilon - \partial_y(\partial_x \mathbf{u}_\varepsilon + \partial_y \mathbf{v}_\varepsilon) = 0 \end{cases}
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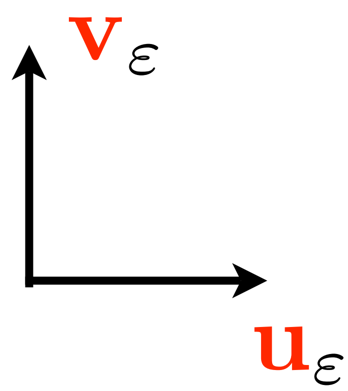

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The problem is well-posed as soon as

$$M_\varepsilon \in W^{1,\infty}(-1,1)$$

A dimensionless model

Scaling

$$\mathbf{u}_\varepsilon(x,y,t) = u_\varepsilon(x,\frac{y}{\varepsilon},t), \qquad \mathbf{v}_\varepsilon(x,y,t) = \varepsilon\, v_\varepsilon(x,\frac{y}{\varepsilon},t)$$

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A quasi-1D model, non local in y

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- Decoupled 1D transport equations

$$(\partial_t + M\partial_x)^2\, u = \partial_x^2 \left[E(u)\right]$$

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Towards the well-posedness analysis

$$u(x, y, t) \quad \xrightarrow{\mathcal{F}_x} \quad \mathbf{u}(k, y, t)$$

$$\boxed{\mathbf{U}(\mathrm{k,y,t}) = \Big(\ \mathbf{u}(\mathrm{k,y,t}), \ \big[(\partial_t + ikM)\mathbf{u}\big](k, y, t) \Big)^t}$$

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First order evolution problem: $\dot{\mathbf{U}} + ik\mathbf{A}(M)\mathbf{U} = 0$

where $\mathbf{A}(\mathrm{M})$ is the operator in $\mathrm{L}^2(-1,1)^2$

$$\mathbf{A}(\mathrm{M}) = \begin{pmatrix} M & I \\ E & M \end{pmatrix}$$

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Intuitively, one expects well-posedness if and only if

$$\sigma\big(\mathbf{A}(M)\big) \subset \mathbb{C}^- \quad \big(\ \mathbb{C}^- := \{\mathcal{I}m\ z \leq 0\}\ \big).$$

General properties of $\mathbf{A}(\mathrm{M})$

Let $S(u,v) = (v,u)$, then one has

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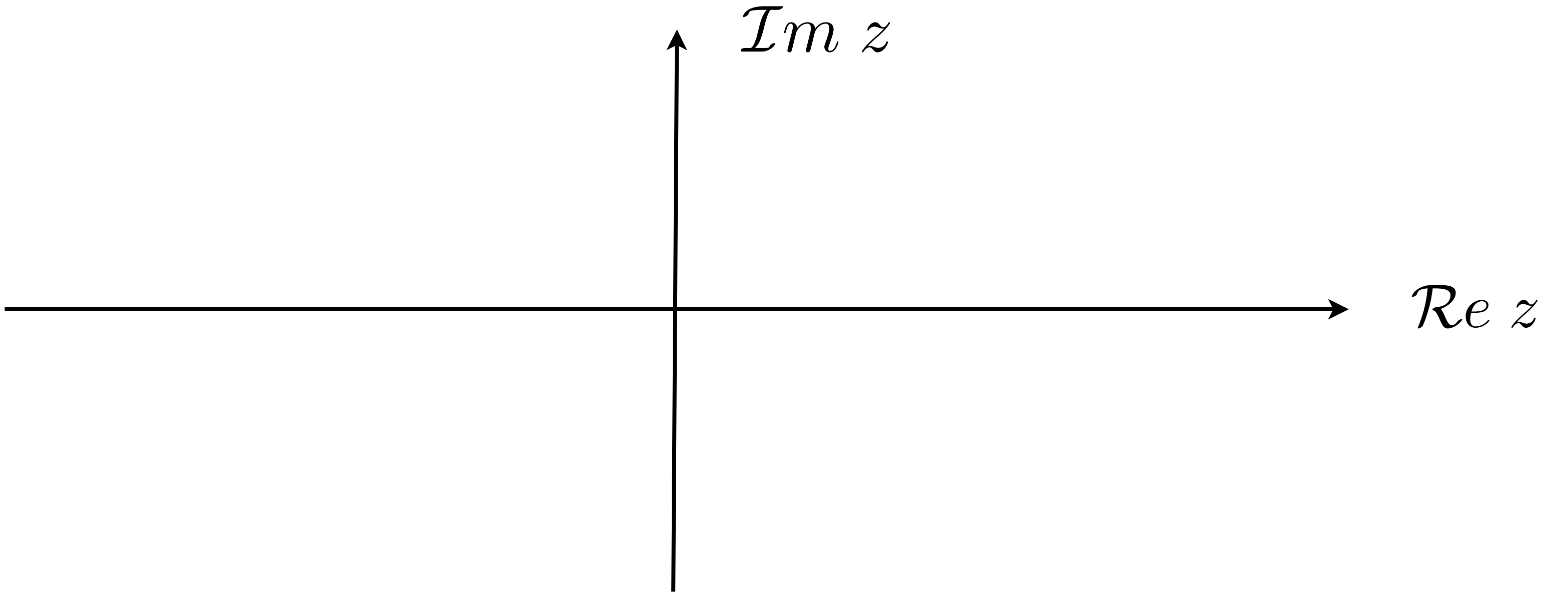
The operator $\mathbf{A}(\mathrm{M})$ is a compact perturbation of

$$\mathbf{A}_0(M) = \begin{pmatrix} M & I \\ 0 & M \end{pmatrix}$$

Structure of the spectrum of $\mathbf{A}(\mathrm{M})$



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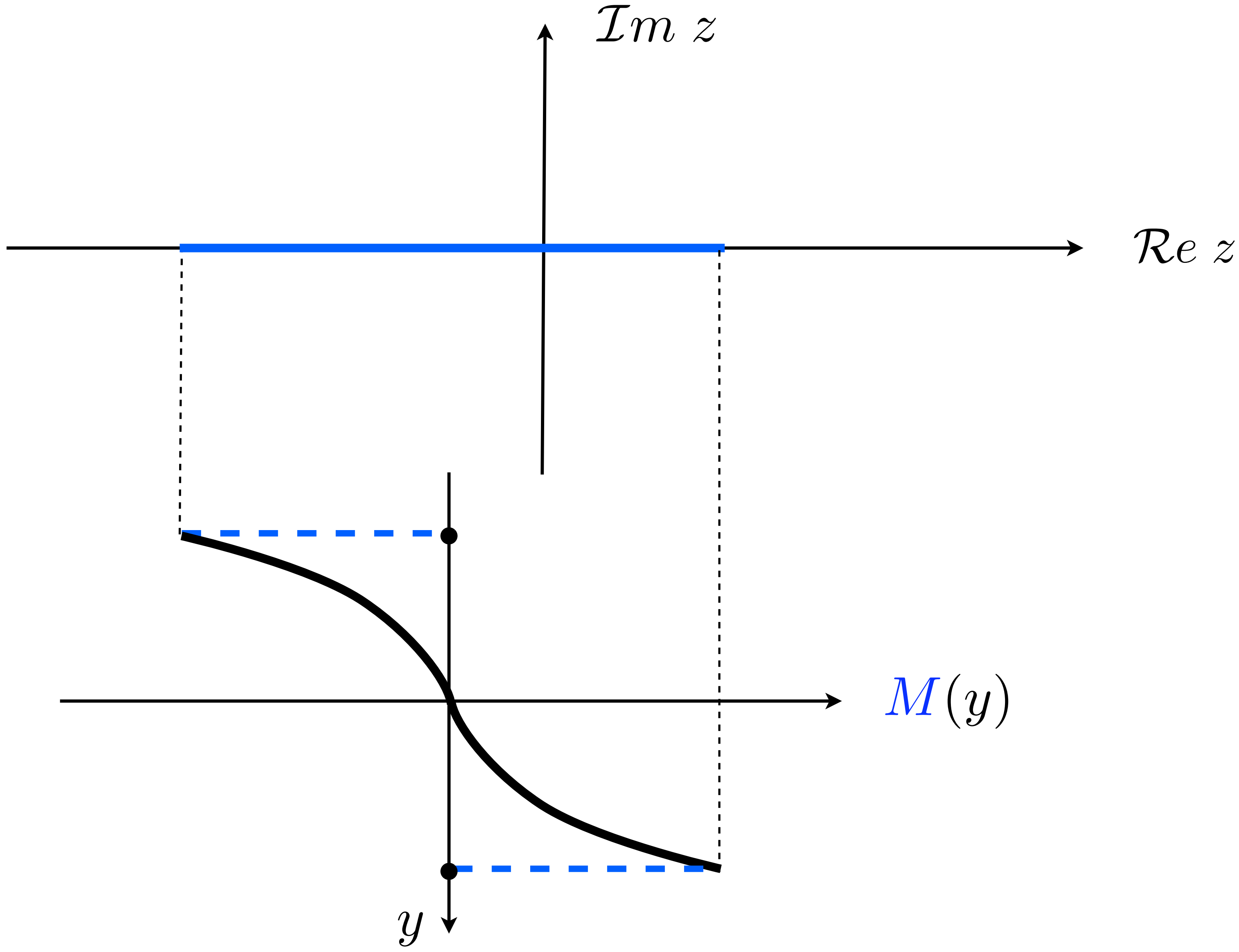

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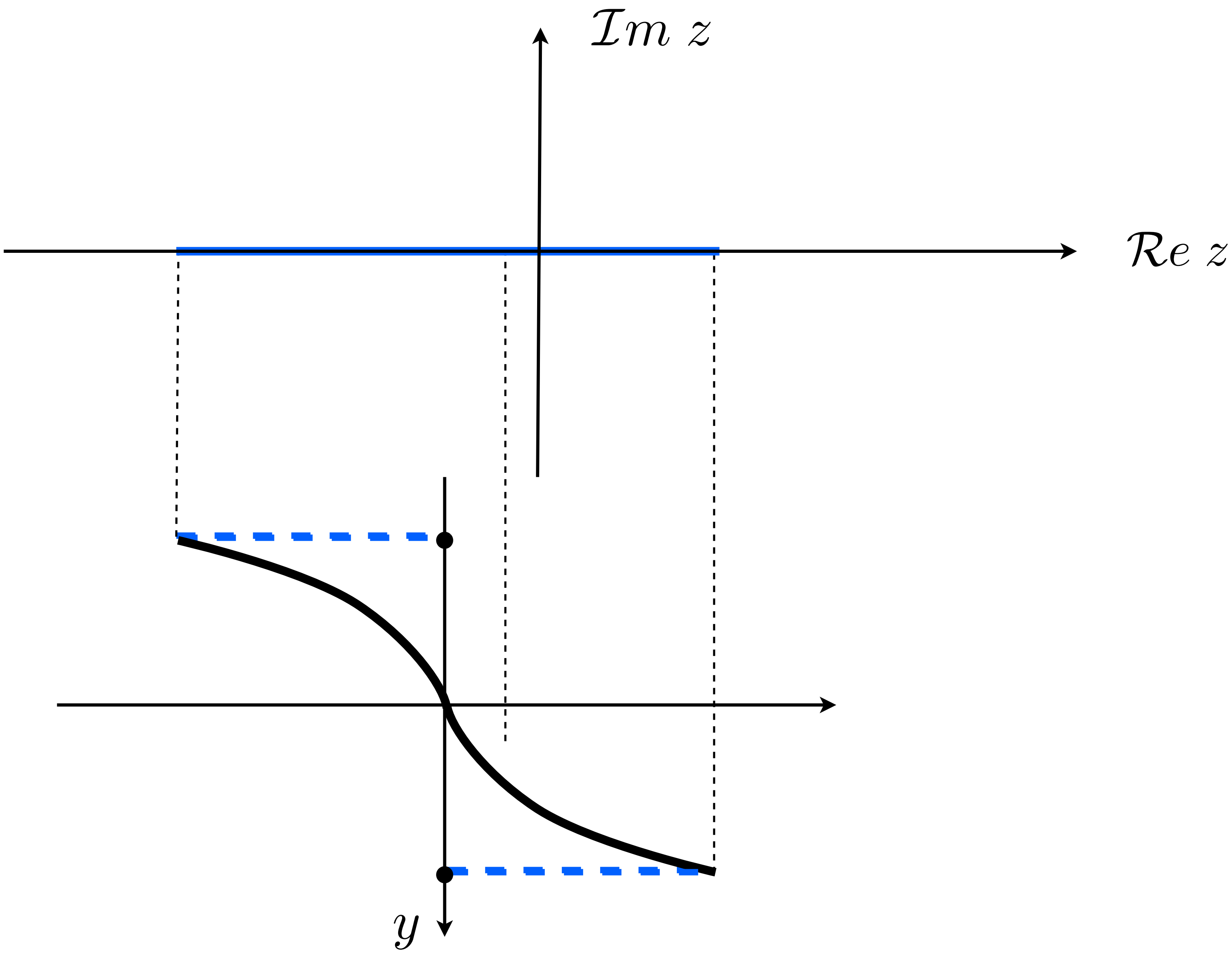

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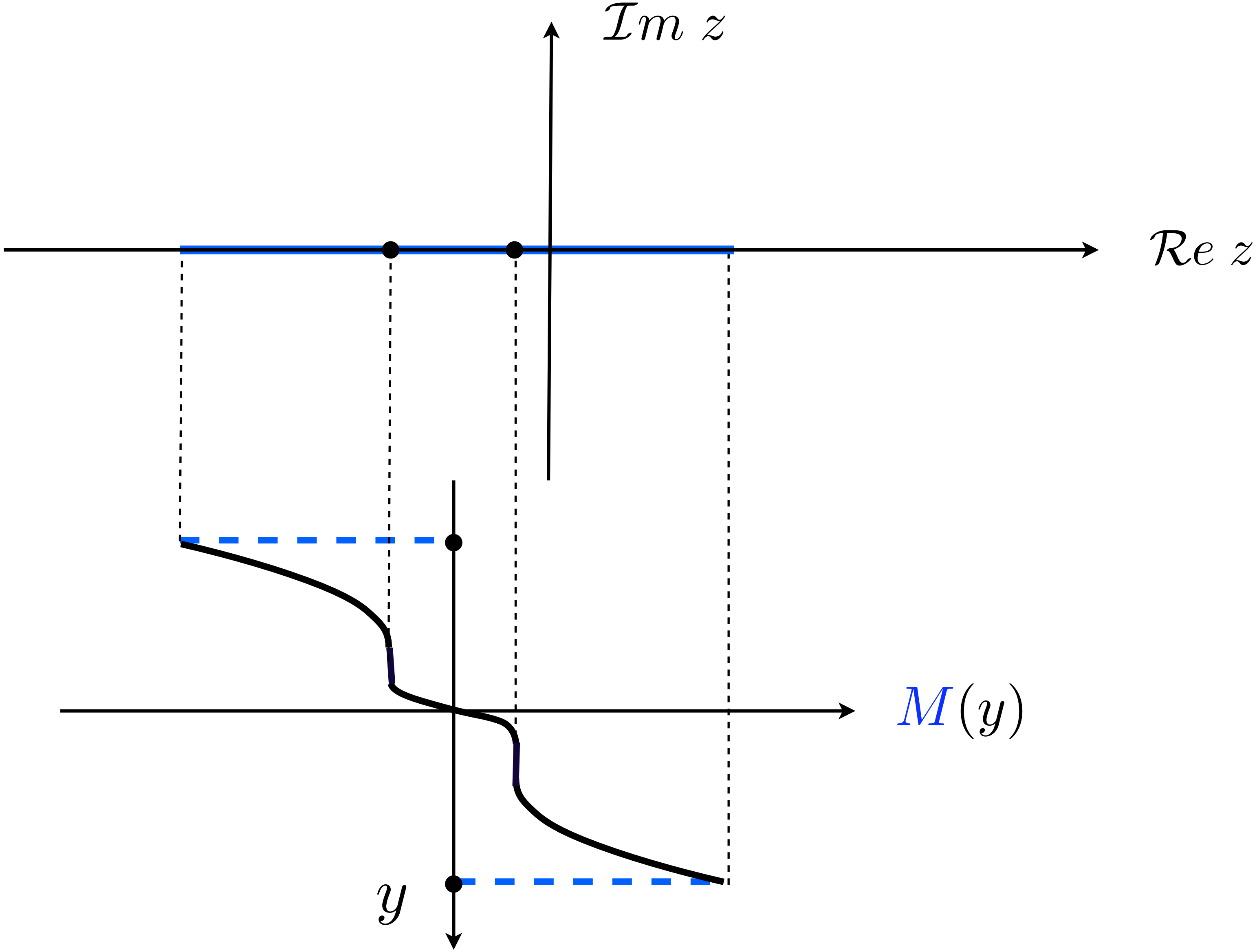

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Lemma : A number $\lambda \in \mathbb{C} \setminus \mathbf{Im}\, M$ is an eigenvalue of $\mathbf{A}(\mathrm{M})$ if and only if:

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This eigenvalue is simple associated with

$$(\mathbf{u}_\lambda, \dot{\mathbf{u}}_\lambda) = \left(\frac{1}{(\lambda - M)^2}, \frac{1}{(\lambda - M)} \right)$$

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$$\mathbf{A}(\mathrm{M})\ \mathrm{U} = \lambda\ U \iff \begin{cases} M\ u + v = \lambda\ u \\ E(u) + M\ v = \lambda\ v \end{cases}$$

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$$= 0 \iff F_M(\lambda) = 2$$

$$\implies \quad E(u)\left[E\left(\frac{1}{(\lambda - M)^2} \right) - 1 \right] = 0$$

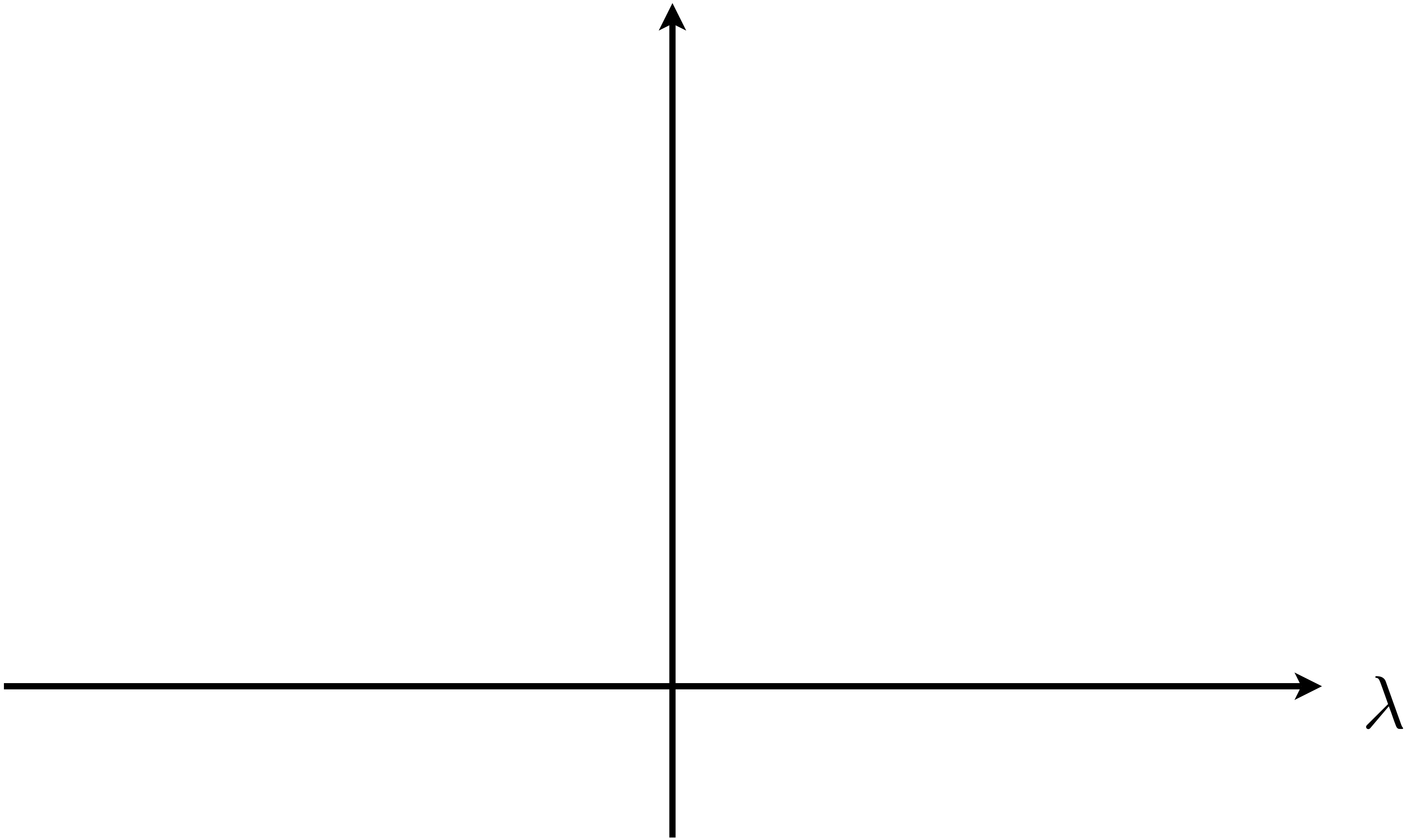
Eigenvalues of $\mathbf{A}(\mathbb{M})$ (2)

The study of real eigenvalues is easier because $F_M(\lambda)$ is real-valued along the real axis

Lemma : The operator $\mathbf{A}(\mathbb{M})$ has exactly two real eigenvalues outside the interval $[M_-, M_+]$

$$\lambda_- < M_- < M_+ < \lambda_+$$

$$F_{M}(\lambda) := \int_{-1}^{1} \frac{dy}{\big(\lambda - M(y)\big)^2}$$



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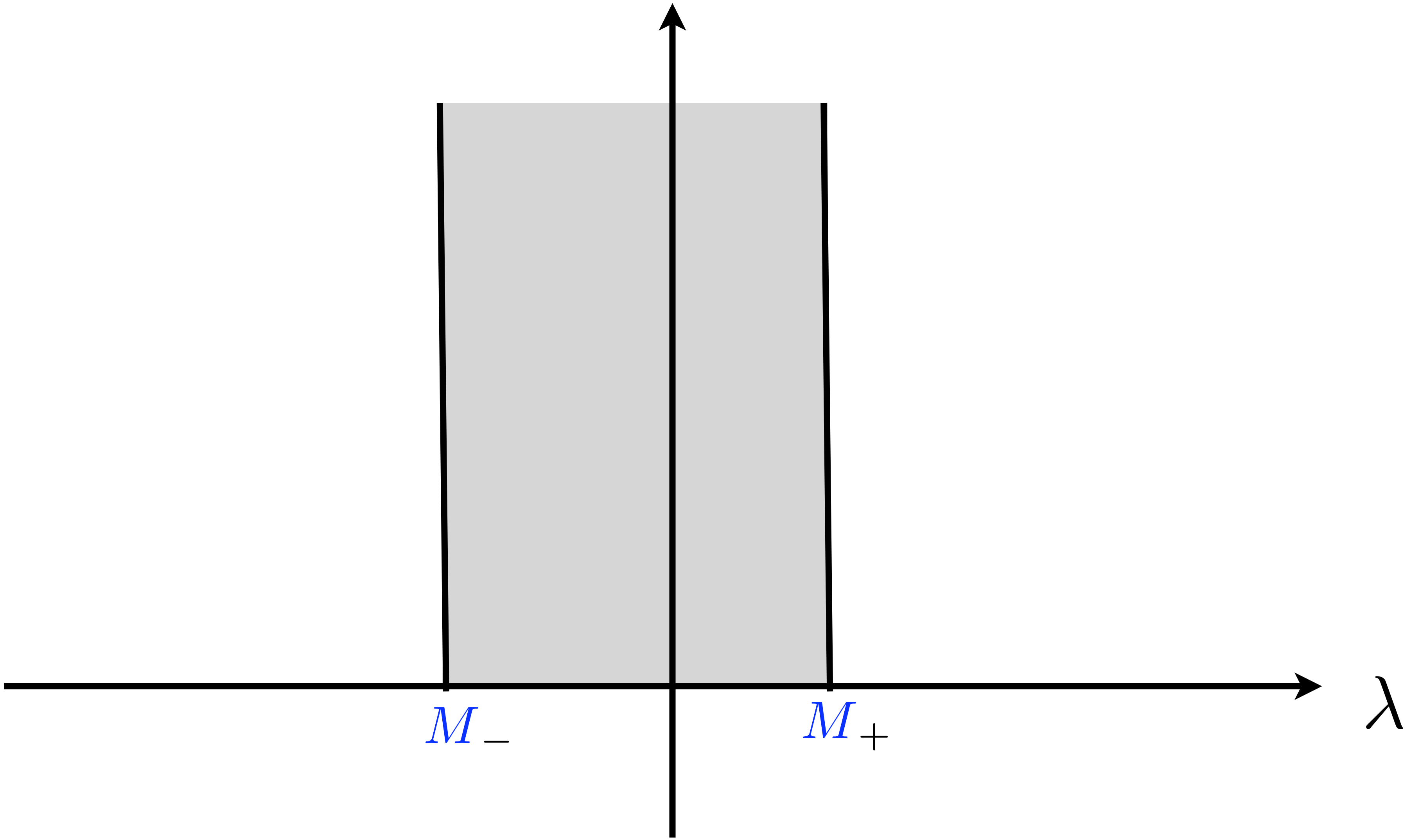

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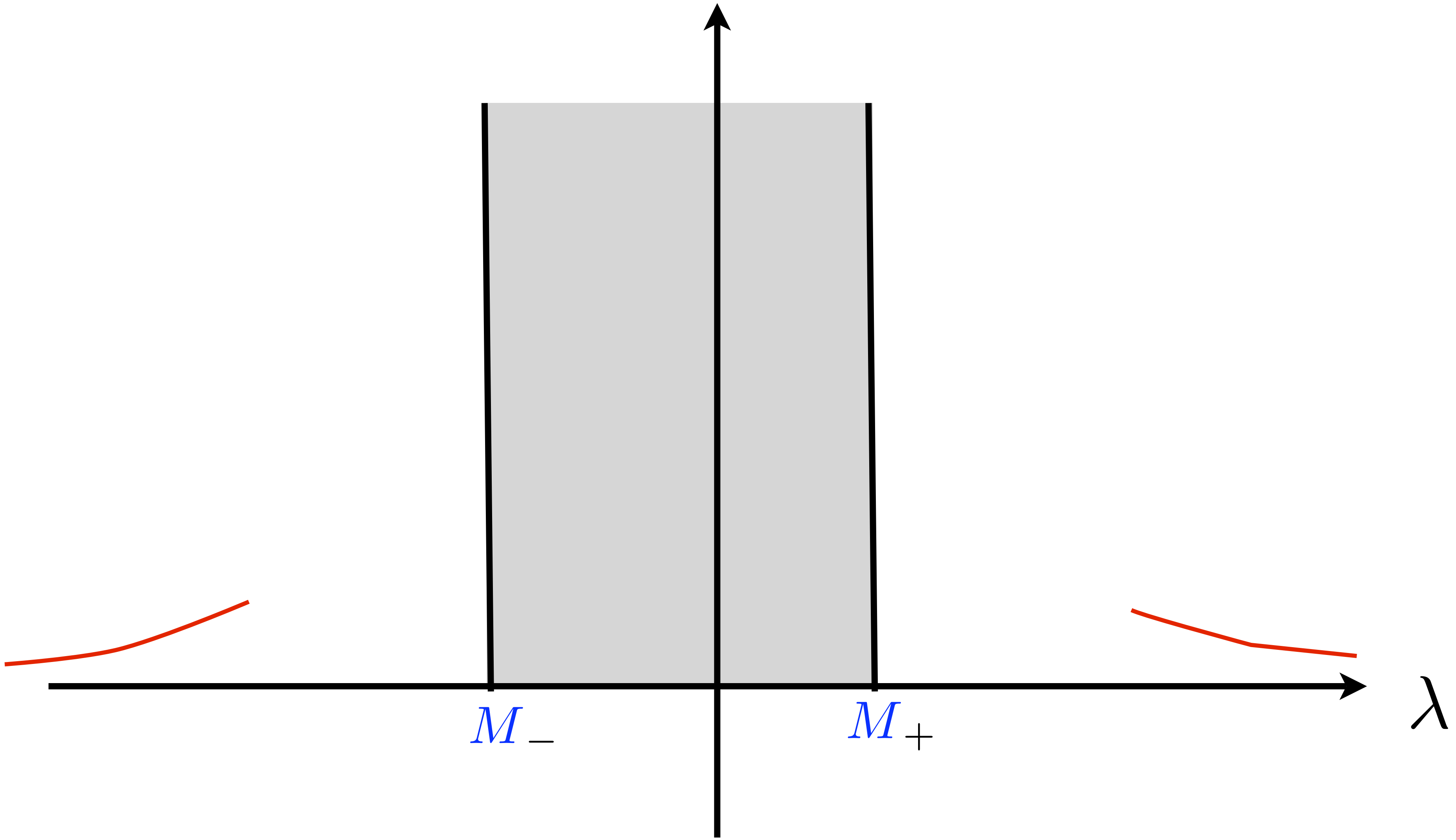

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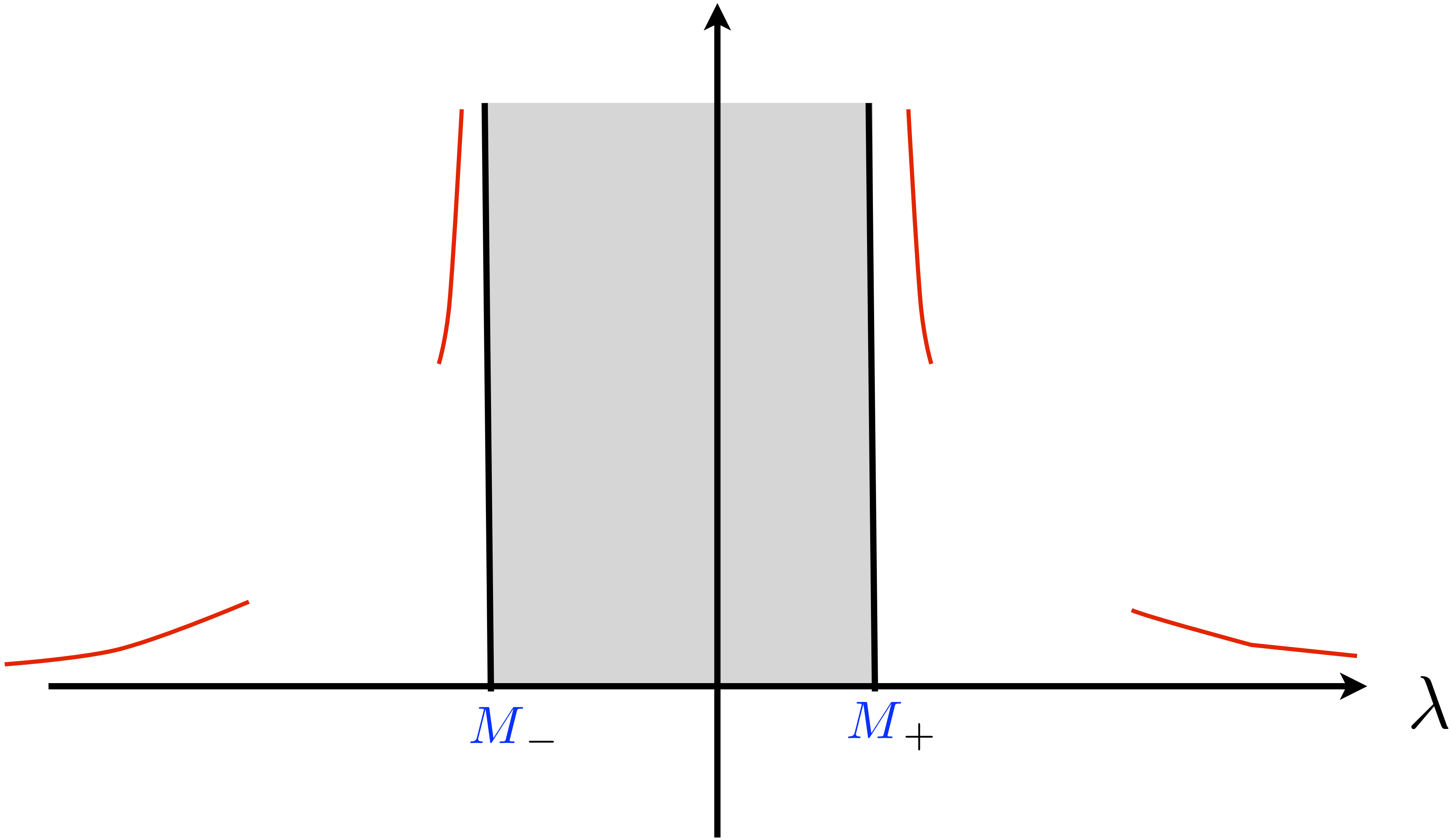

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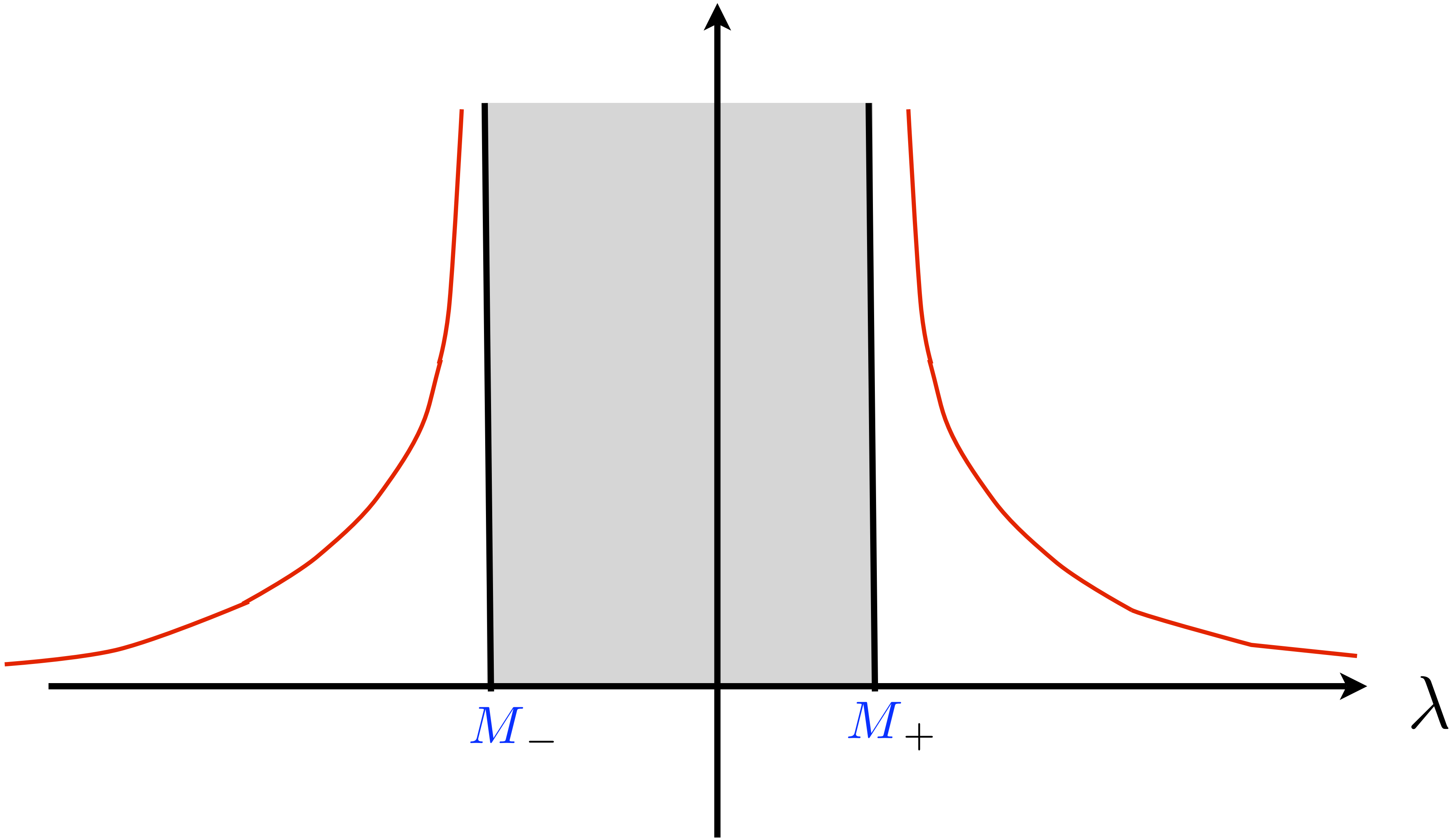

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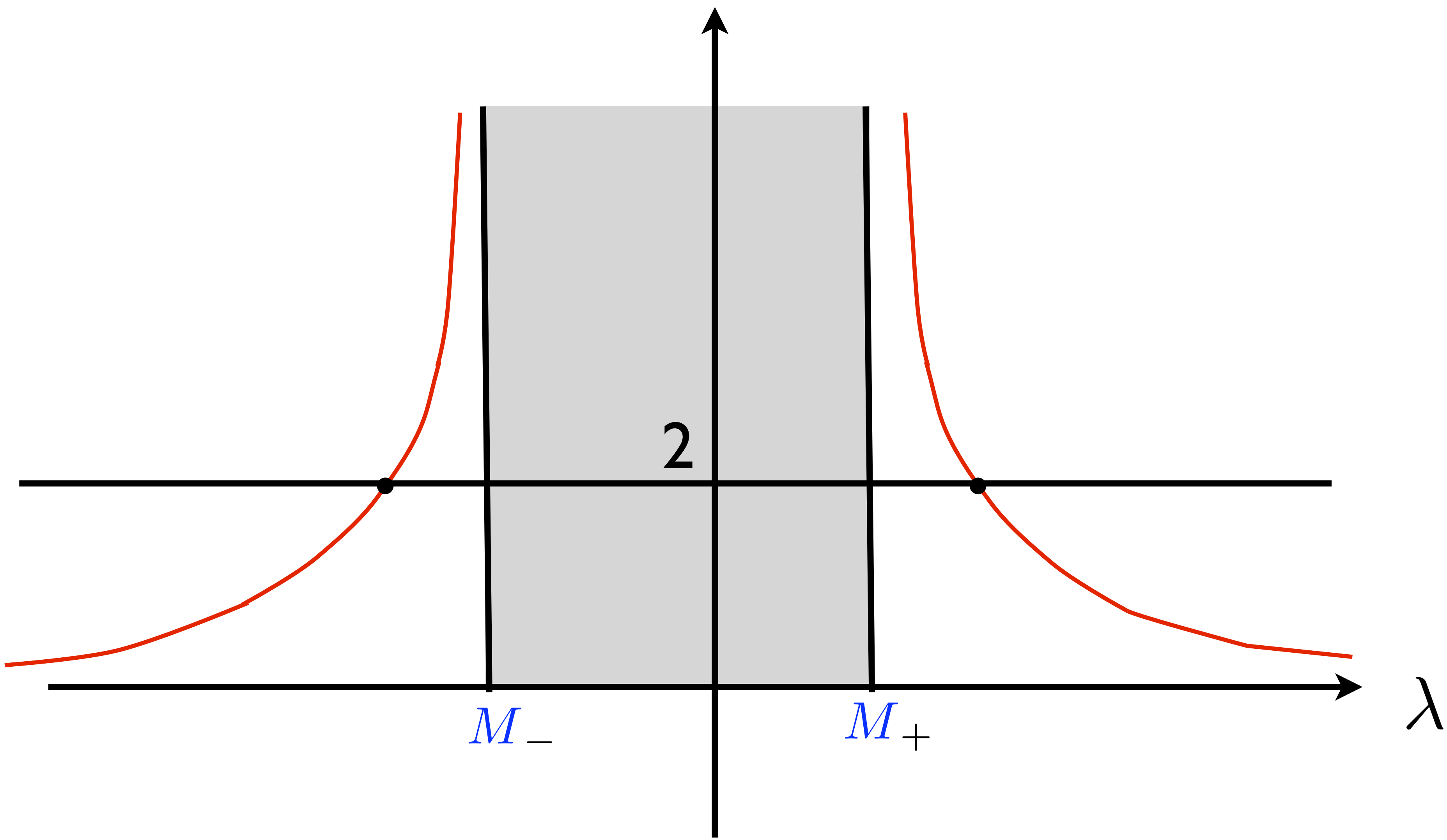

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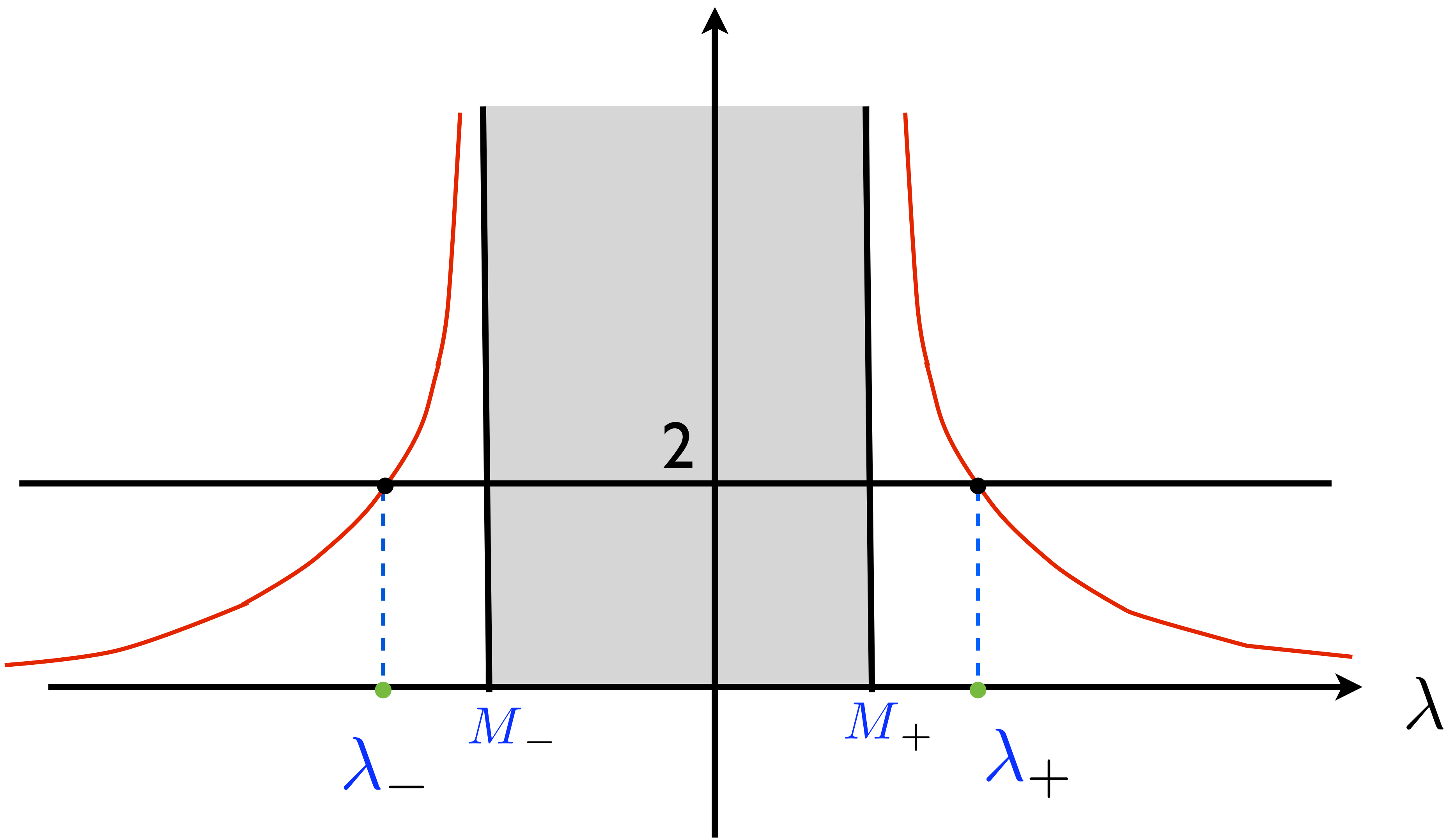
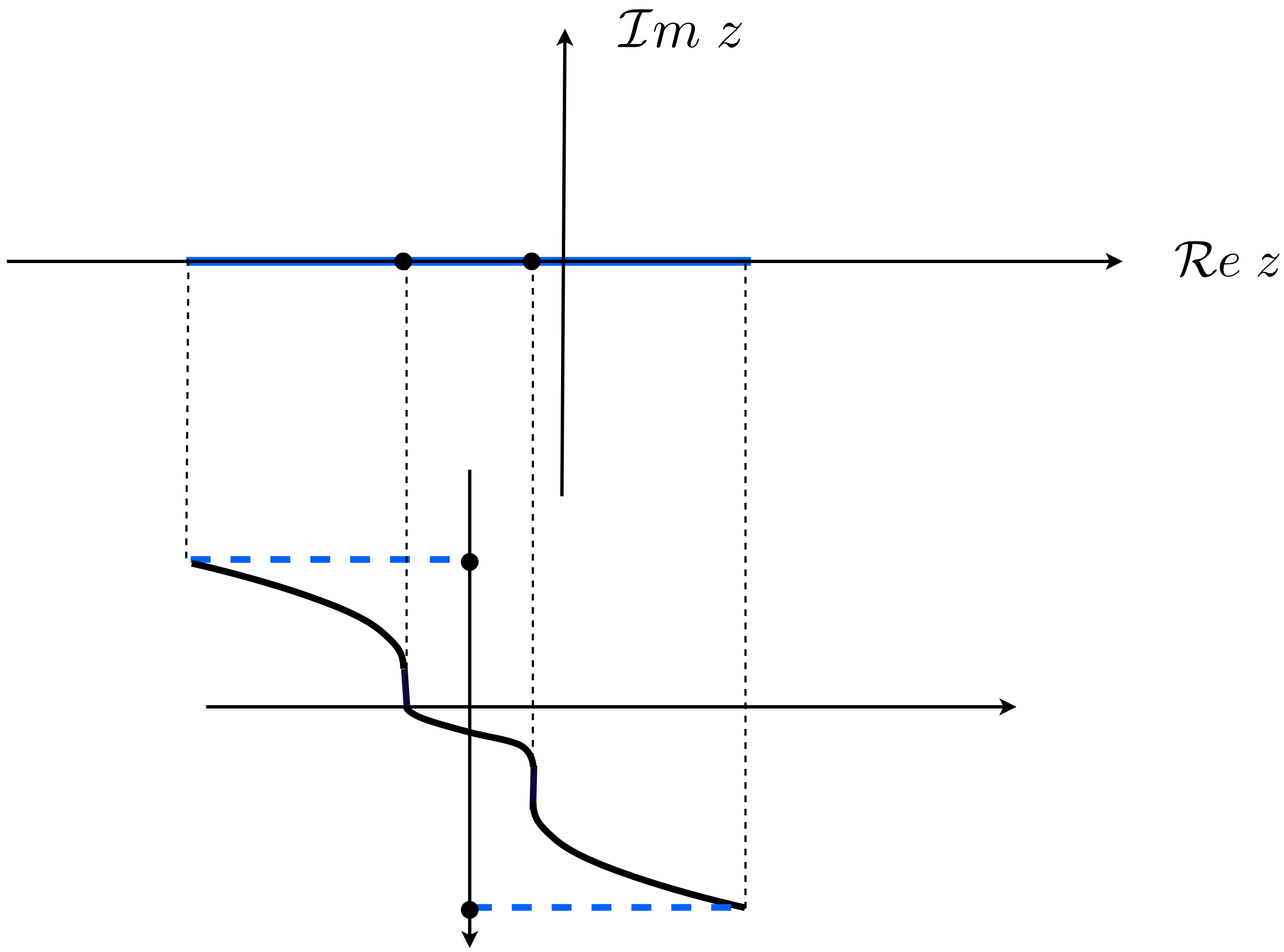


Back to the spectrum of $\mathbf{A}(\mathrm{M})$



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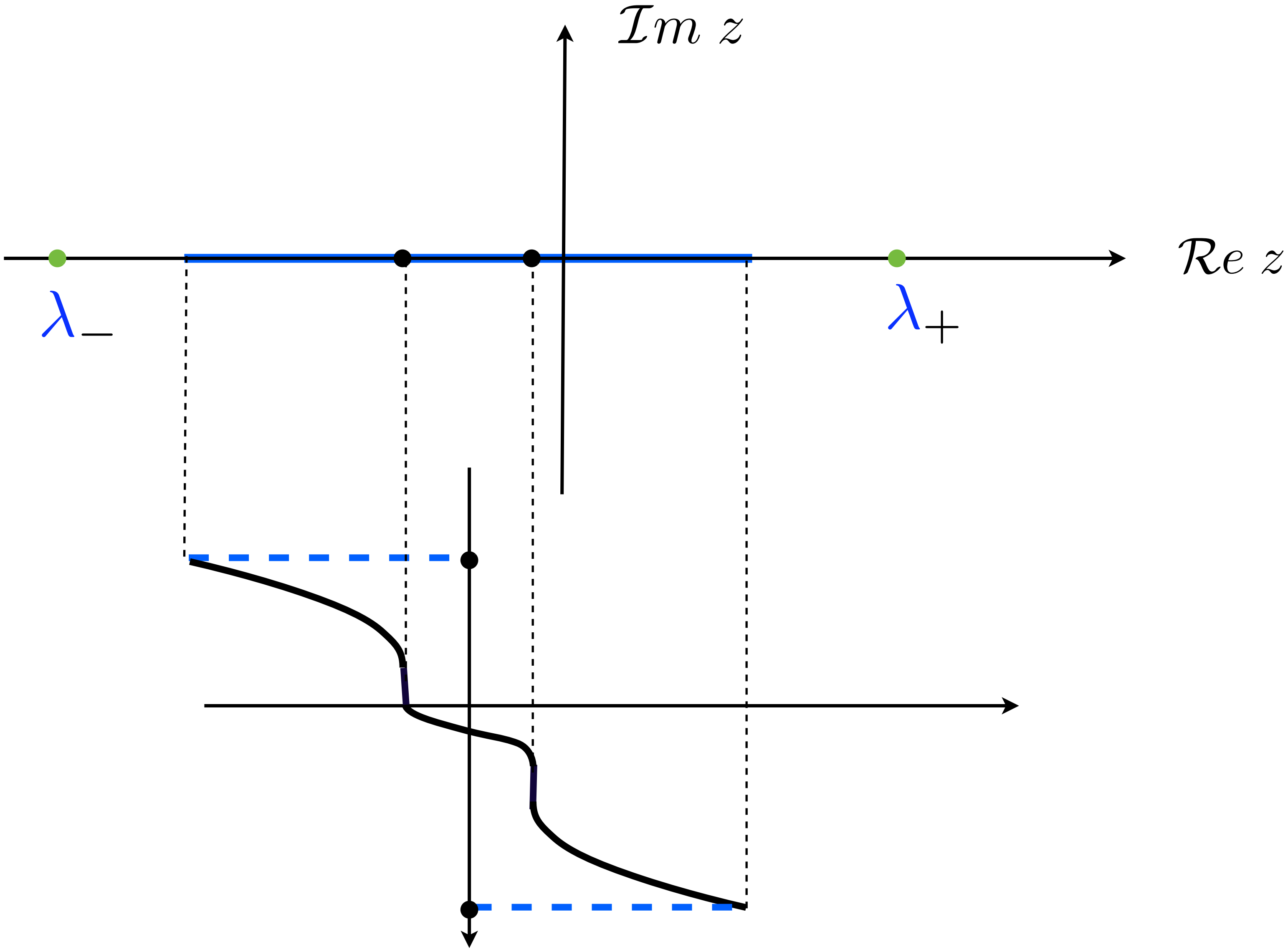

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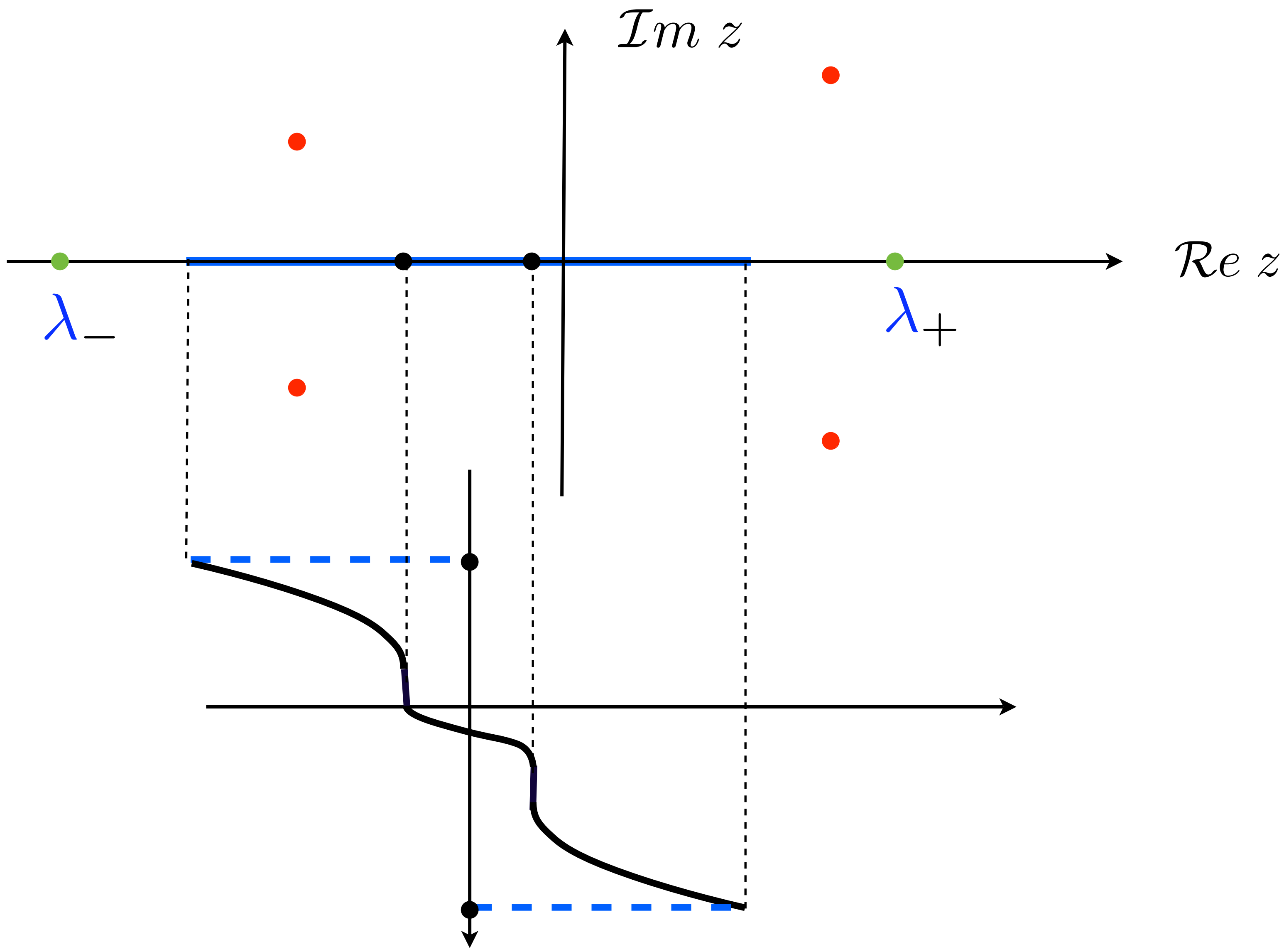

Definition of a stable profile

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$(\mathcal{E})$ has non real solutions

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(*) has been proven in some cases (see later)

A by-product : hydrodynamic instabilities

Theorem : if M is **unstable**, $(\mathcal{P})_\varepsilon$ is **unstable**, i. e.

$$(\mathcal{P})_\varepsilon \left\{ \begin{array}{r} (\partial_t + M\partial_x)^2 u_\varepsilon - \partial_x(\partial_x u_\varepsilon + \partial_y v_\varepsilon) = 0 \\ \varepsilon^2\,(\partial_t + M\partial_x)^2 v_\varepsilon - \partial_y(\partial_x u_\varepsilon + \partial_y v_\varepsilon) = 0 \end{array} \right.$$

$$\|u_\varepsilon\|_{L^2_x(L^2_y)} + \|v_\varepsilon\|_{L^2_x(L^2_y)} \geq C(u_0, u_1)\, e^{\alpha \frac{t}{\varepsilon}}$$

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These are new results for hydrodynamic instabilities in compressible fluids, proven by perturbation theory

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This is a low freq. approach in opposition to the high freq. approach of O. Laffite & al for Rayleigh -Taylor instability

Stability results

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3. One concludes using perturbation theory for eigenvalue problems (Kato)

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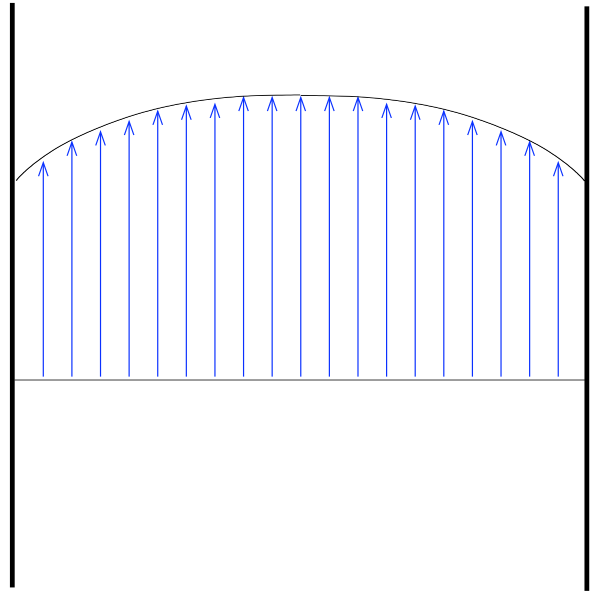

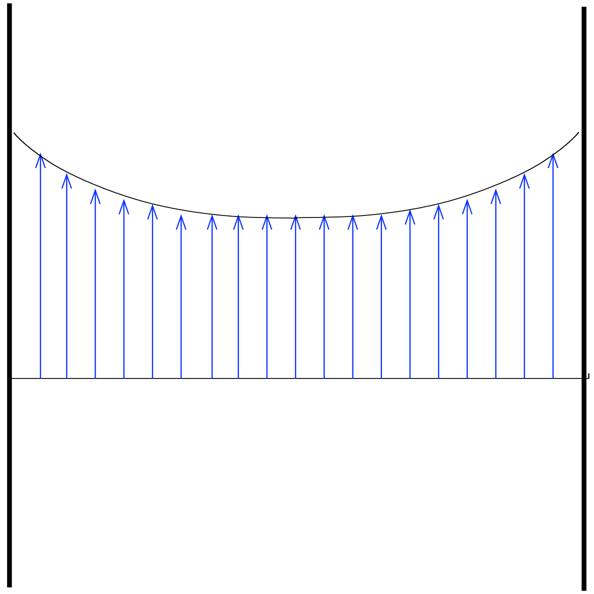

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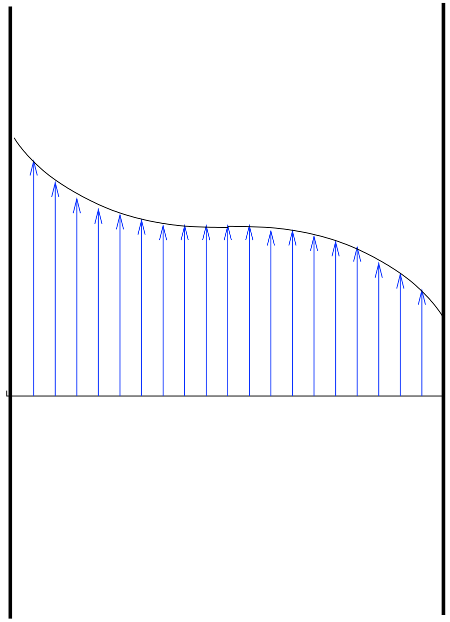

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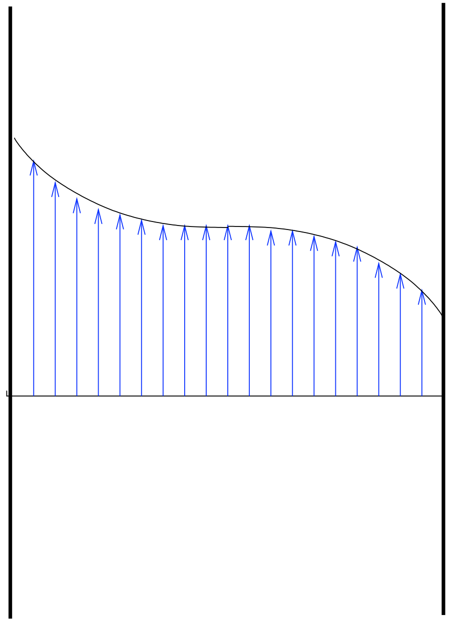

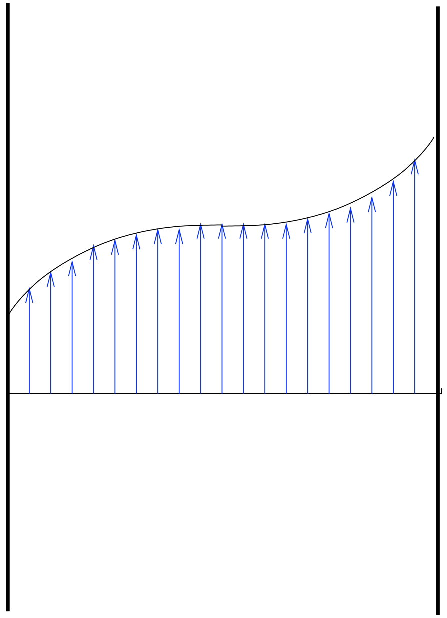

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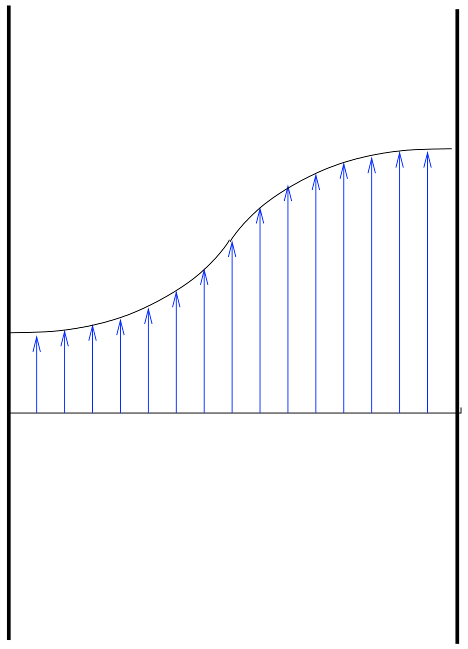

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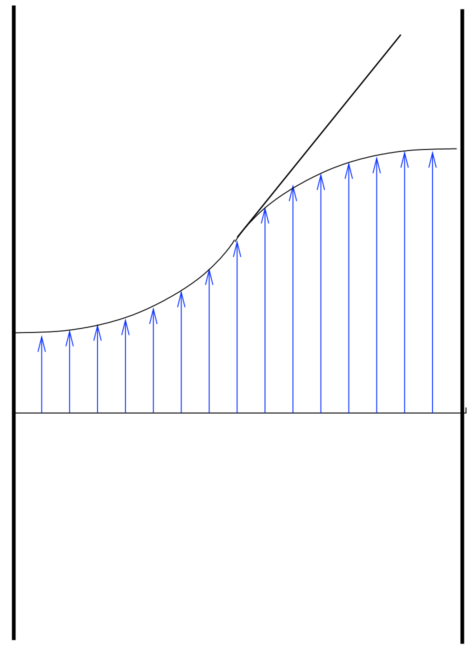

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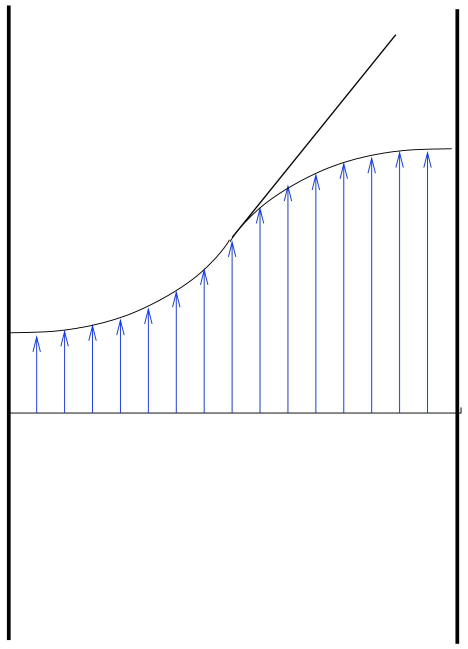

$$M(y)^2 \leq M'(0)^2 \, y^2$$

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Instability results (I)

Application : $M(y) = a\tanh(\alpha\, y), \quad a > 0, \quad \alpha > 0.$

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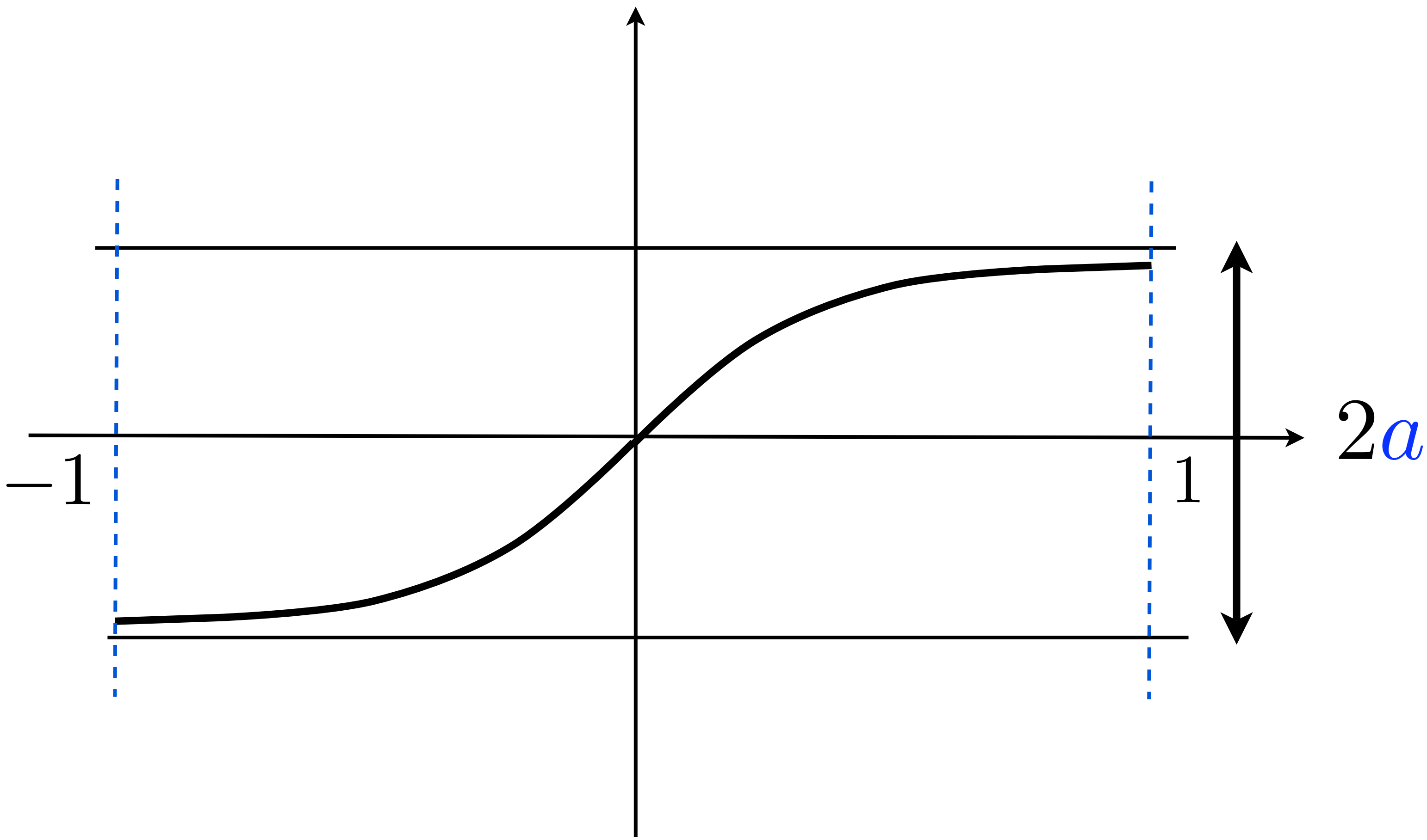

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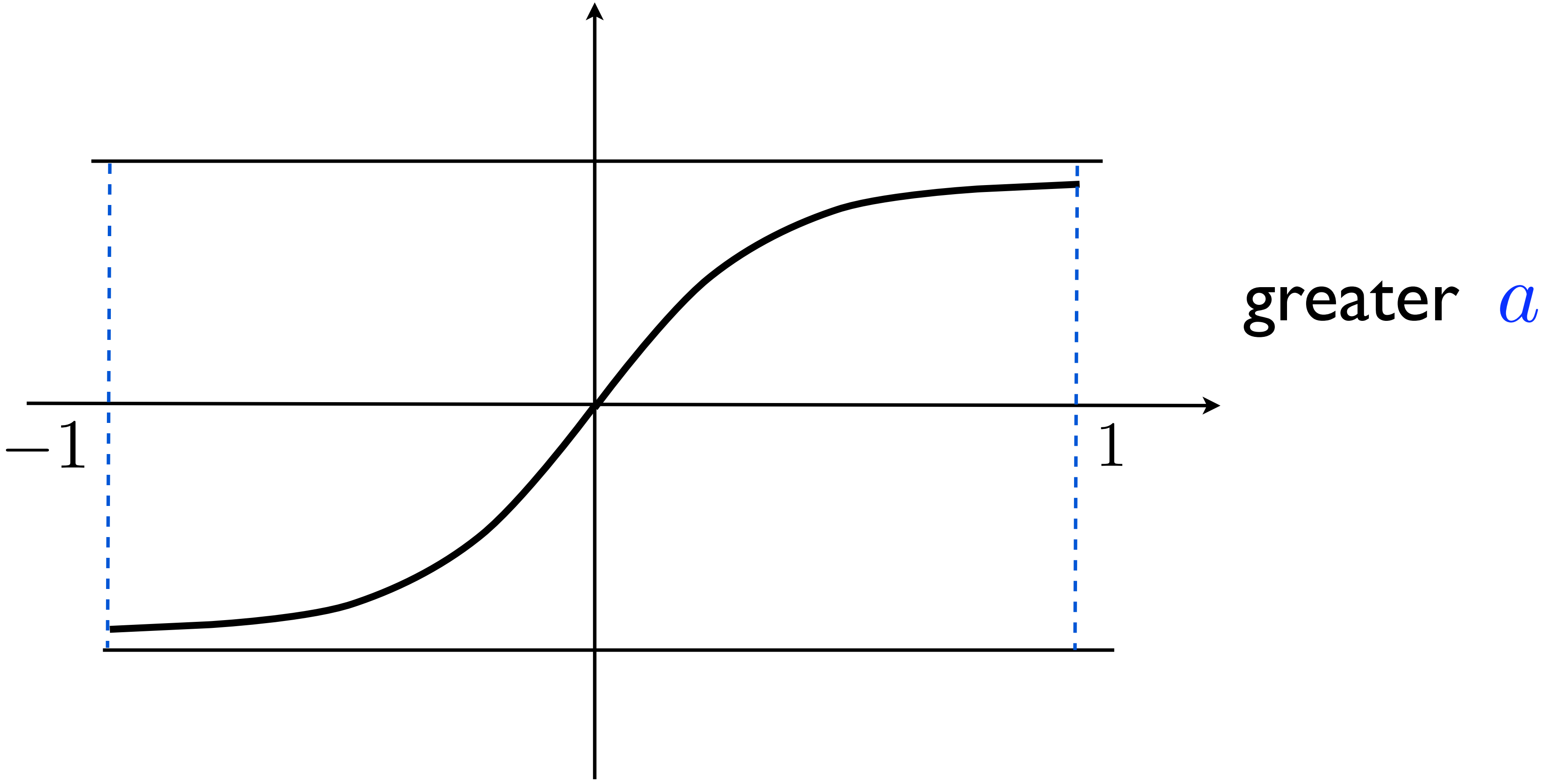

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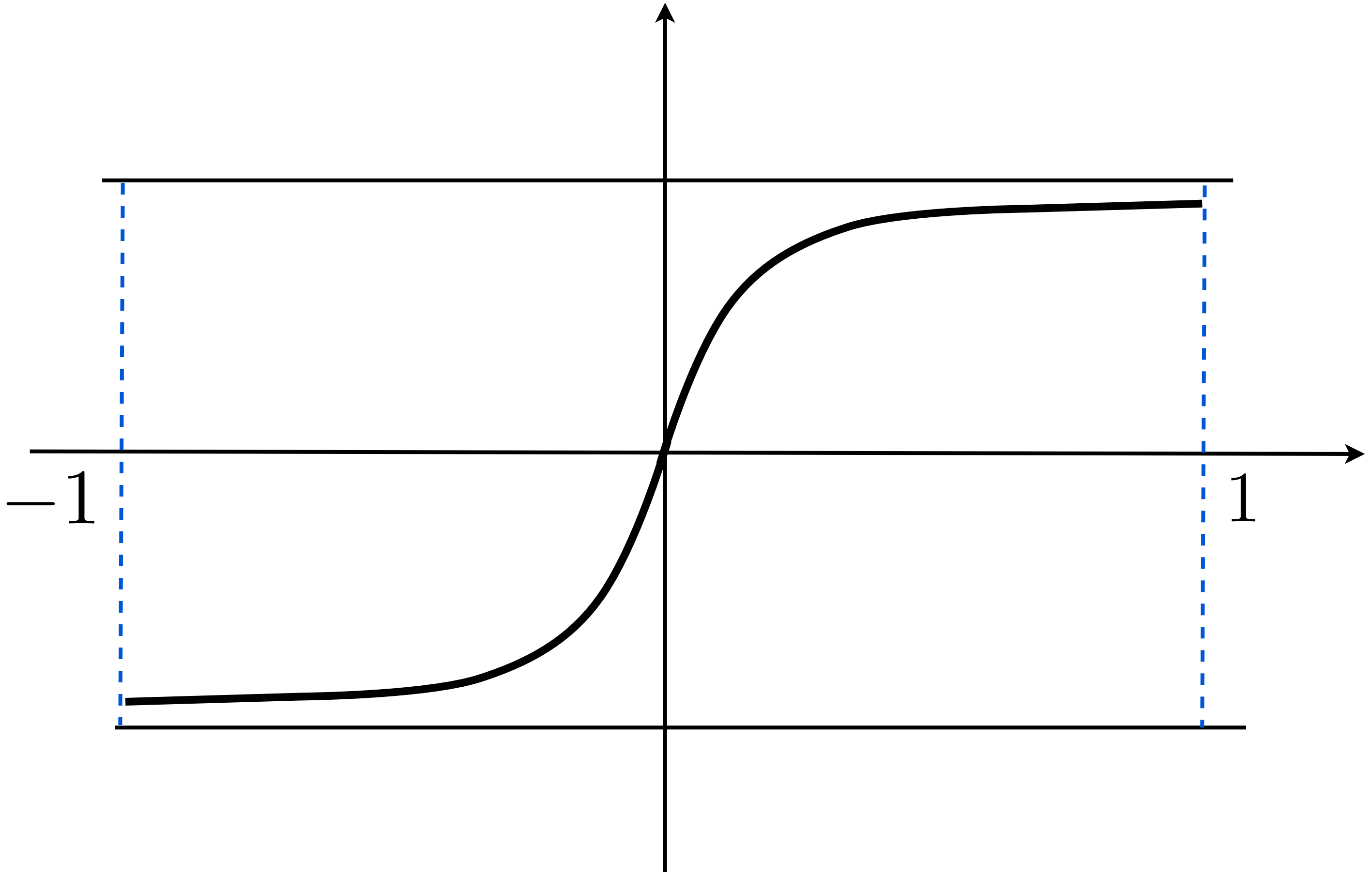

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The profile M is unstable if and only if (*)

$$\alpha > \alpha^* \quad \text{and} \quad a < \left[1 - \alpha \tanh \alpha\right]^{\frac{1}{2}}$$

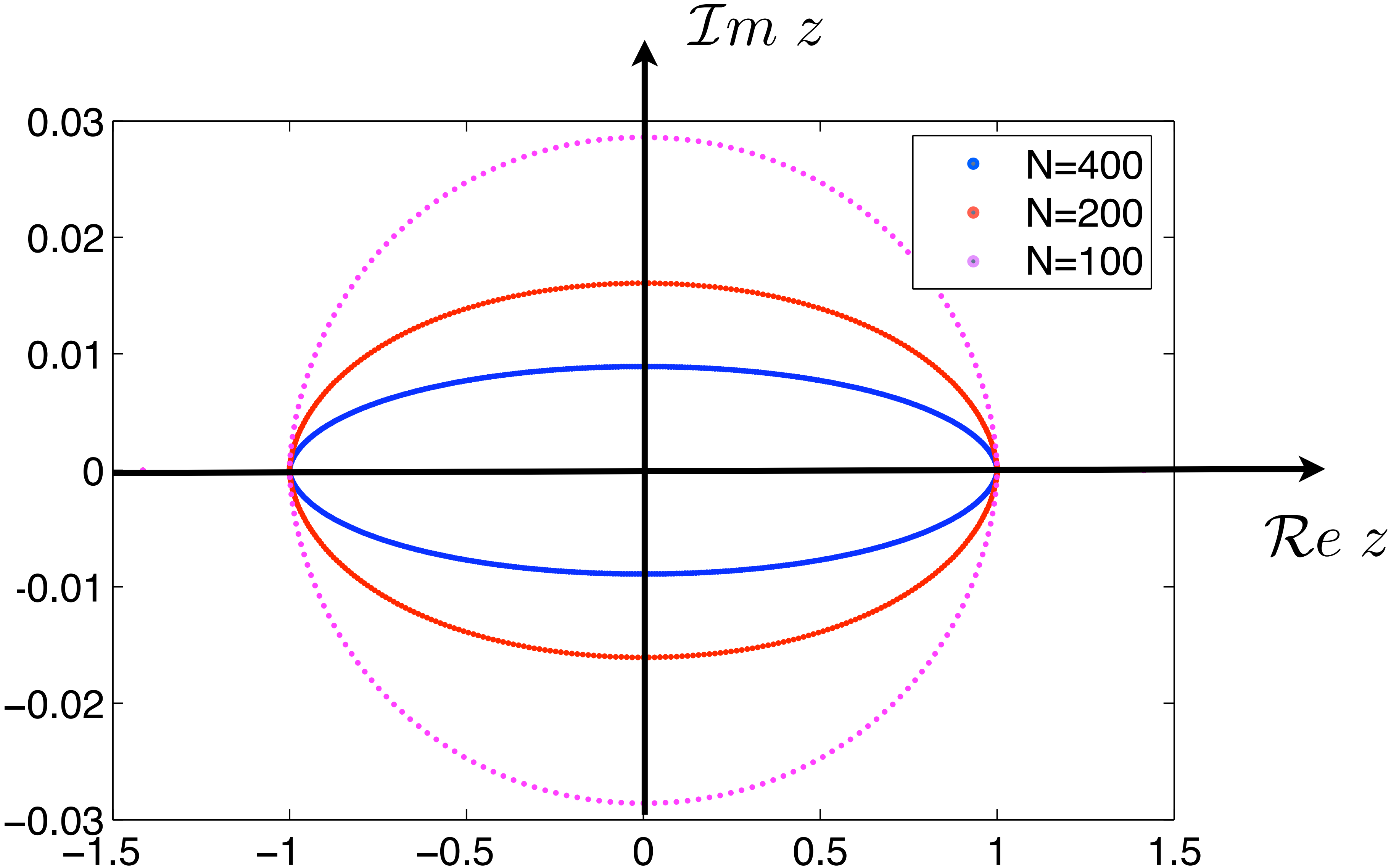
(*) $\quad \alpha > \alpha^* \implies \alpha \tanh \alpha < 1.$

Computation of discrete spectra

With finite dimensional approximation spaces one constructs discrete approximations $\mathbf{A}_h(M)$ of $\mathbf{A}(\mathrm{M})$

One computes the spectrum of $\mathbf{A}_h(M)$

The case of a linear profile $M(y) = y$



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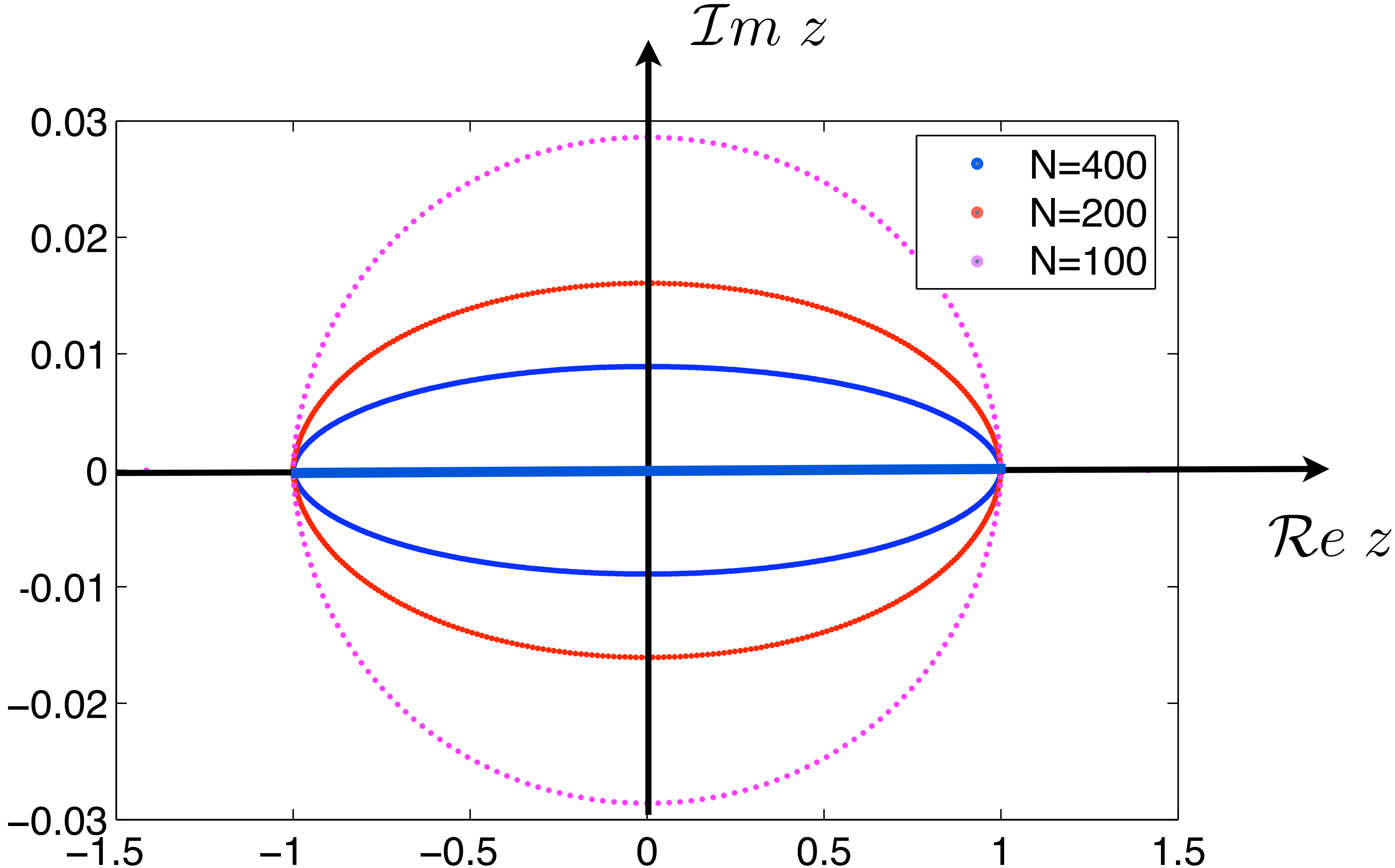

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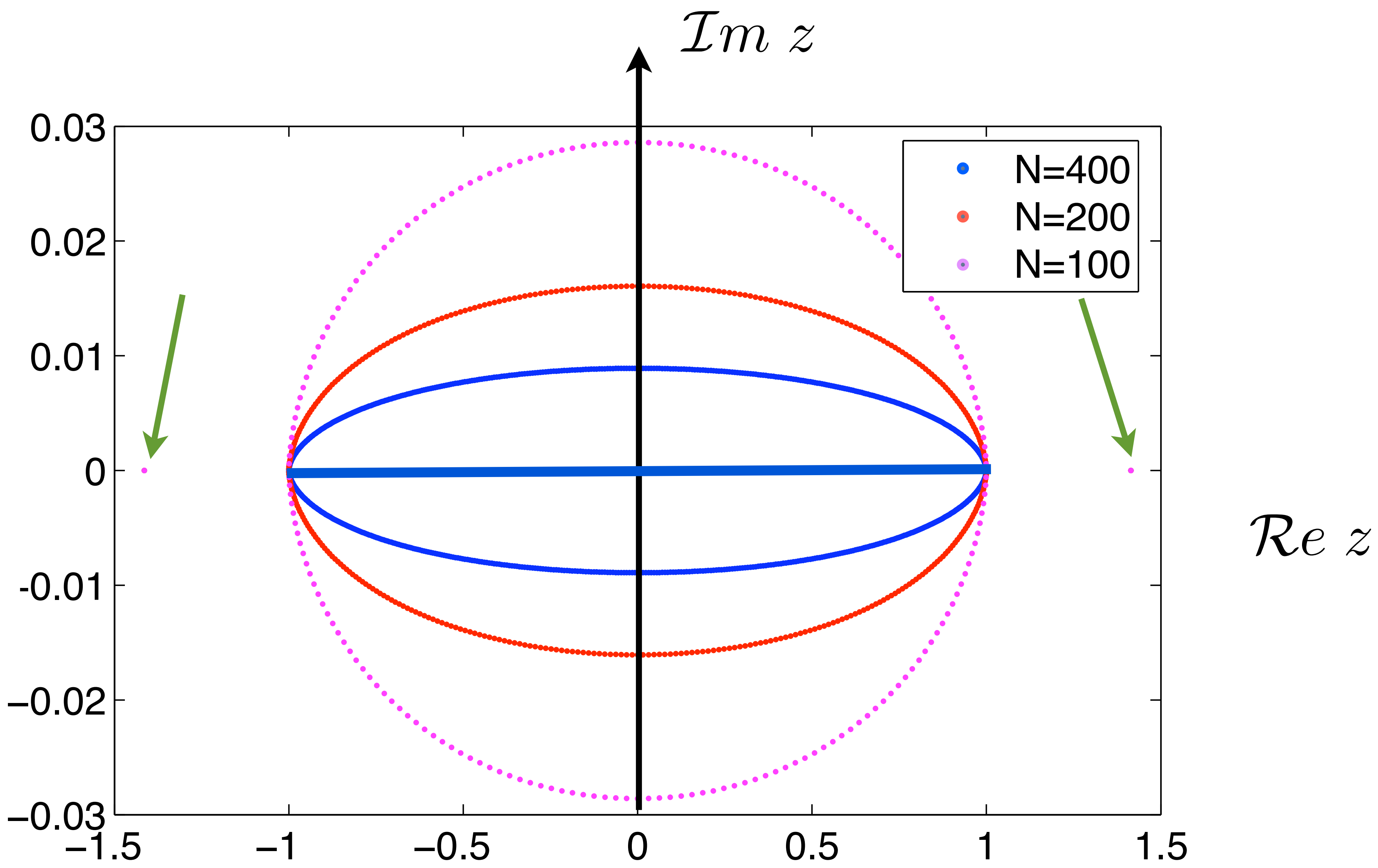

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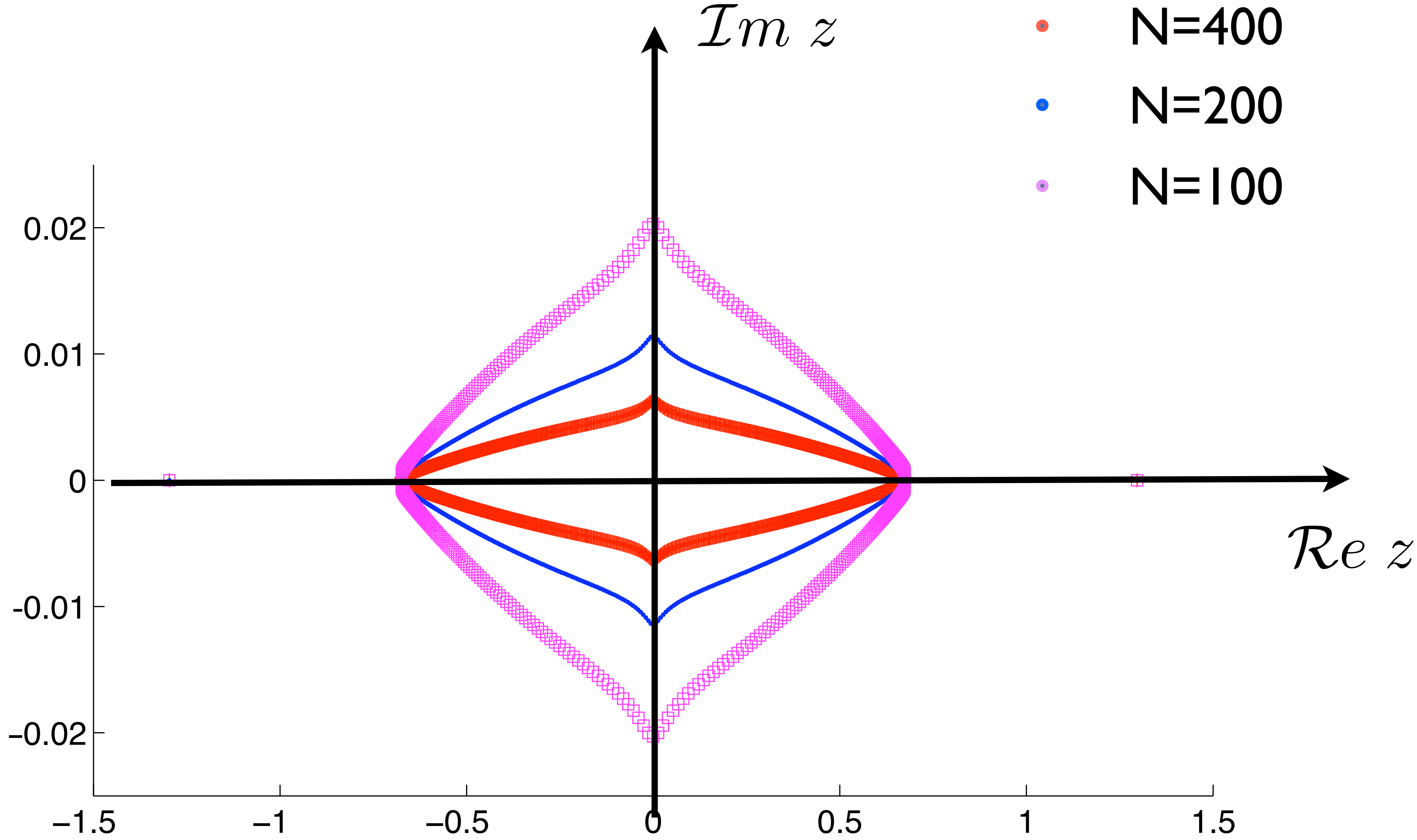

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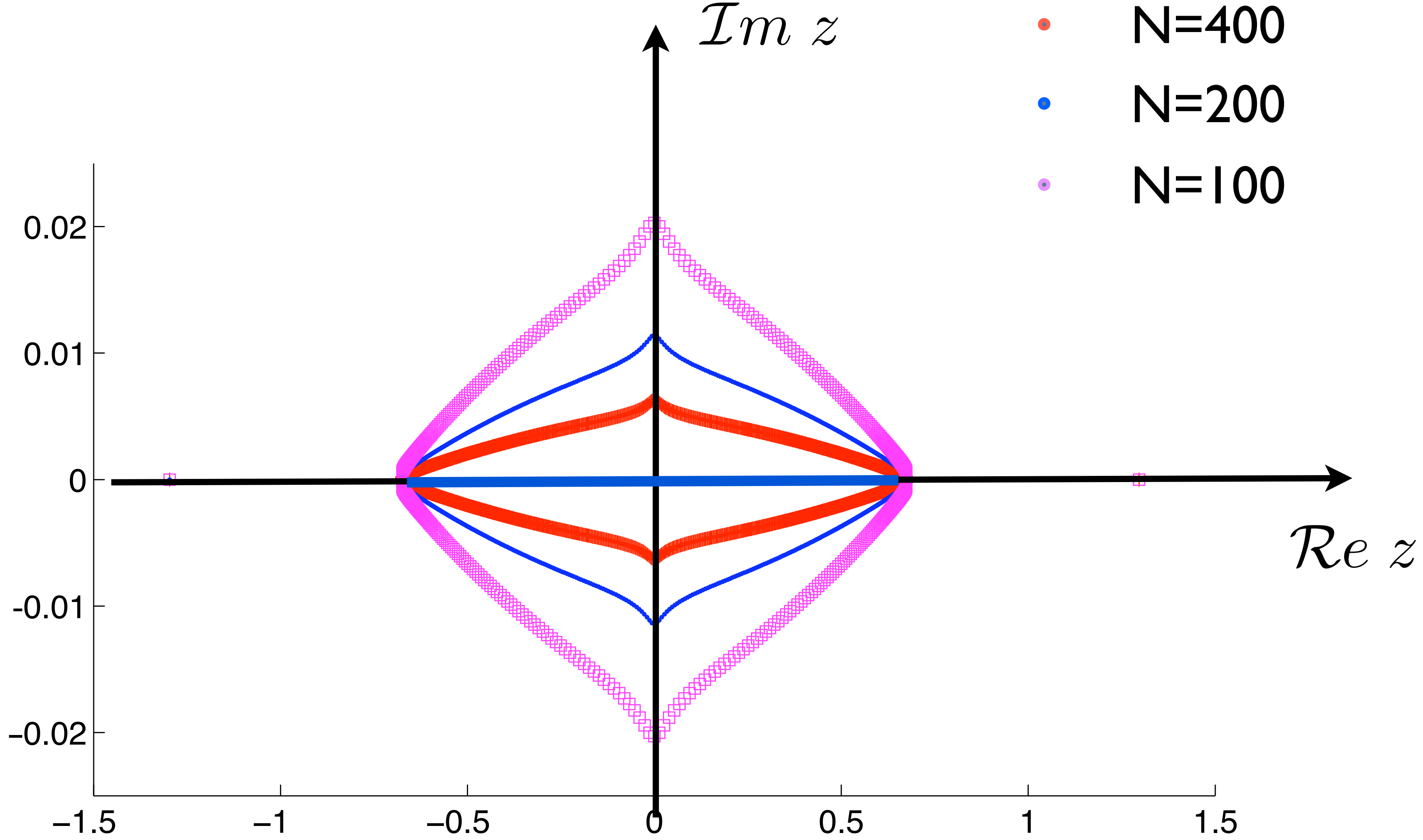

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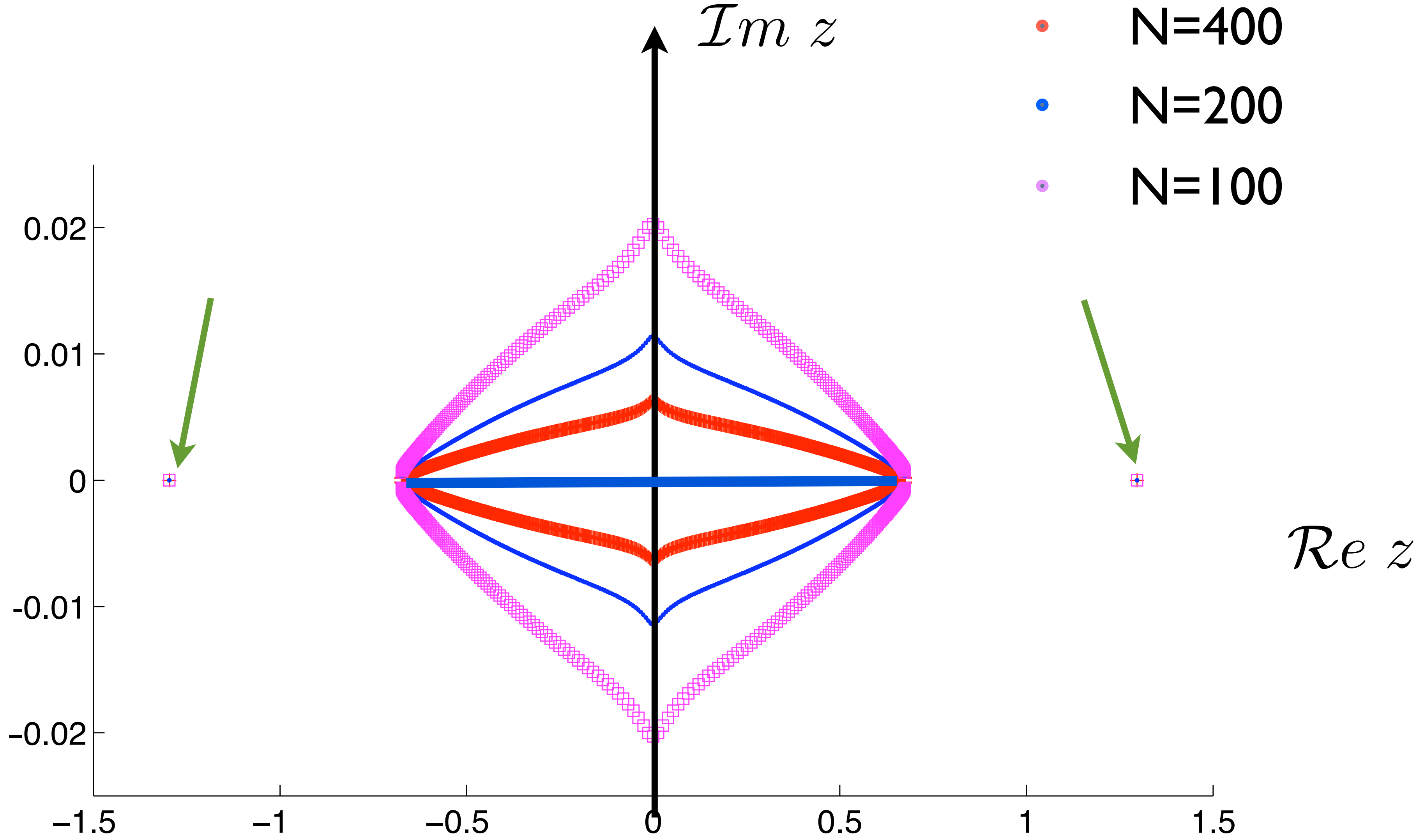

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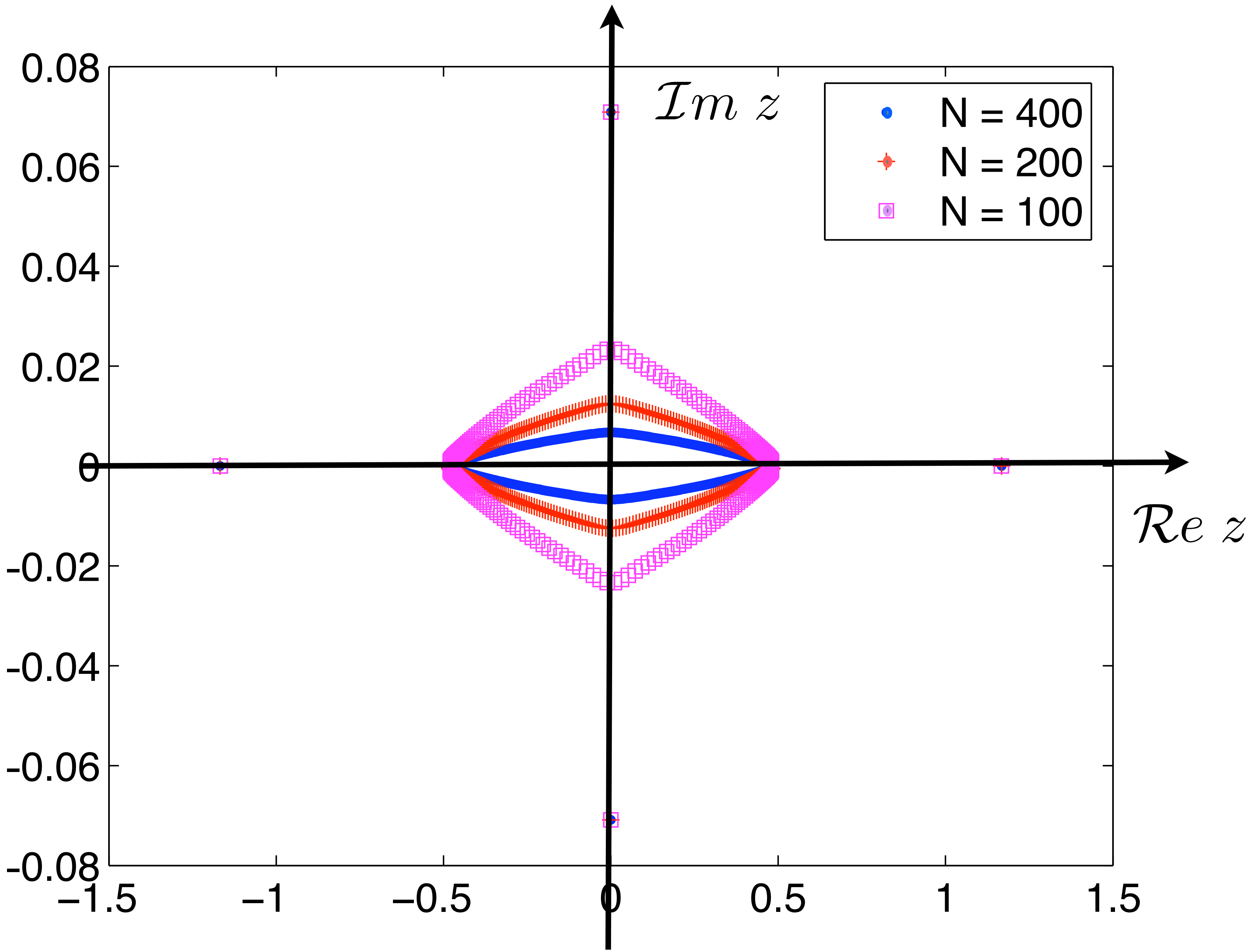

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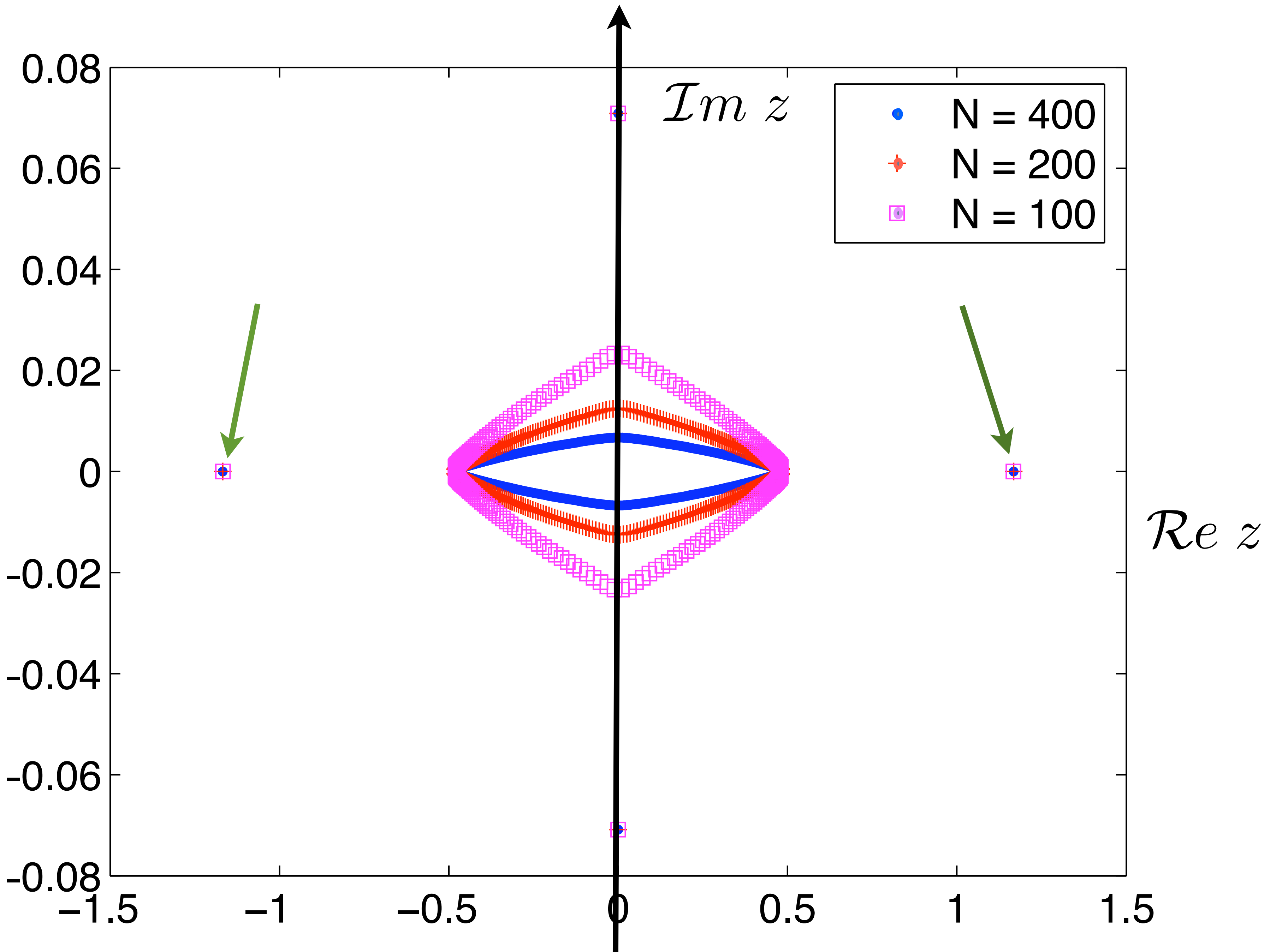

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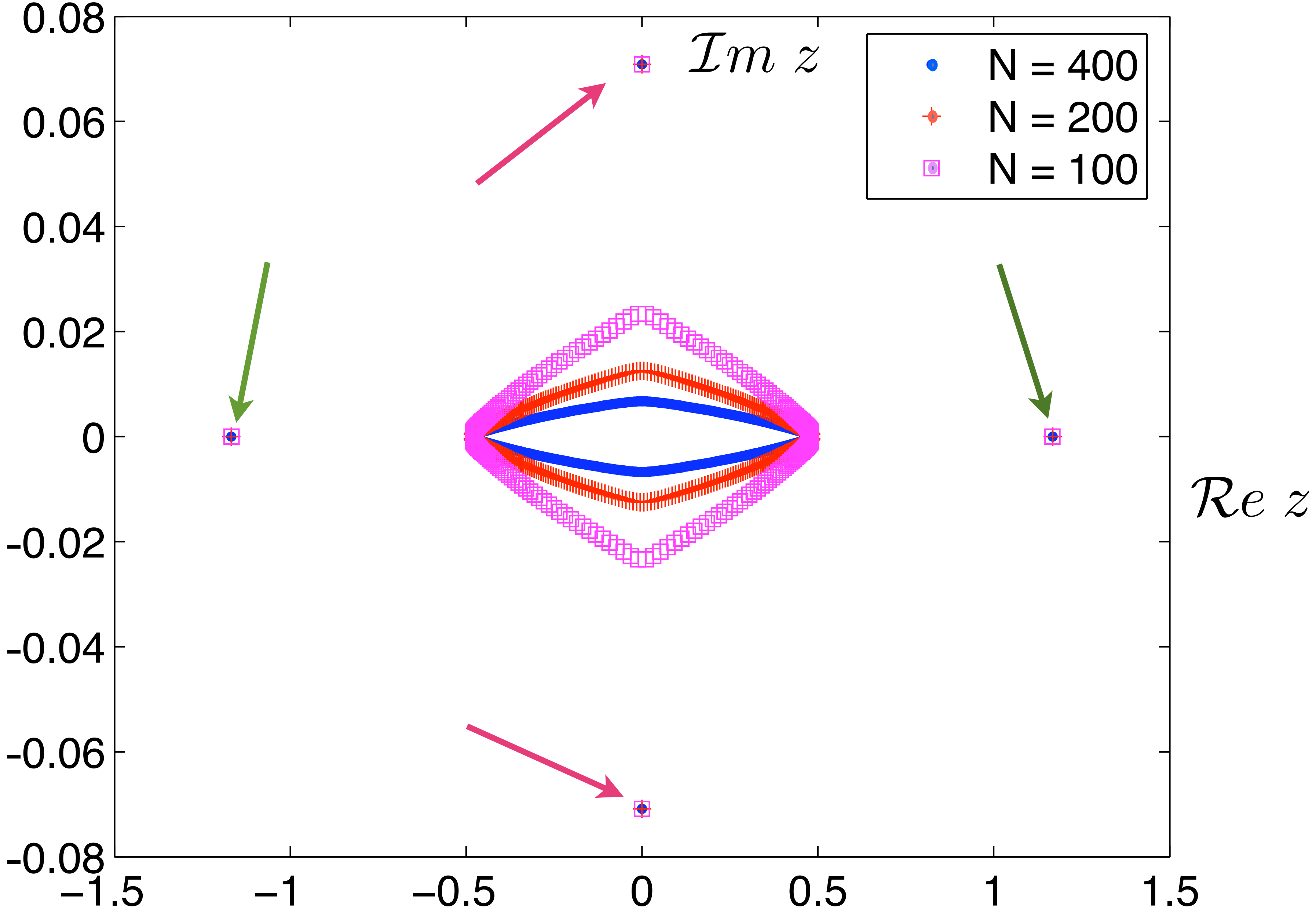

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This points out how delicate is the numerical approximation of the problem

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Theorem : Under assumptions (A) and (B), $(\mathcal{P})$ is weakly well-posed : if $(u_0, u_1) \in H^4_x(L^2_y) \times H^3_x(L^2_y)$, there exists a unique solution

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Proof of the theorem

Since $(\partial_t + M\partial_x)^2\, u = \partial_x^2\big[E(u)\big]$, it suffices to study

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$$U(x,t) \longrightarrow \widehat{\mathbf{U}}(k,\omega), \quad k \in \mathbb{R}, \quad \omega \in \mathbb{C}$$

one obtains an expression of the form :

$$\widehat{\mathbf{U}}(k,\omega) = \frac{\phi(\lambda,k)}{2 - F_M(\lambda)}, \quad \omega = \lambda\, k$$

where ϕ is known explicitly from (u_0, u_1)

Proof of the theorem

$$\phi(\lambda, k) = E\Big(f_0(\cdot,\lambda)\,\widehat{u}_0(\cdot,\lambda)\Big) + E\Big(f_1(\cdot,\lambda)\,\widehat{u}_1(\cdot,\lambda)\Big)$$

$$f_0(y,\lambda) = i\,\frac{M(y)}{\big(\lambda - M(y)\big)^2} - i\,\frac{1}{\lambda - M(y)}$$

$$f_1(y,\lambda) = \frac{1}{\big(\lambda - M(y)\big)^2}$$

$\lambda \mapsto \phi(\lambda, k)$ is singular along $[M_-, M_+]$

Proof of the theorem

With inverse Laplace transform in time:

$$\widehat{\mathbf{U}}(k,t) = \int_{\mathcal{I}m\lambda=\lambda_I} \frac{\phi(\lambda,k)}{2-F_M(\lambda)}\, e^{-ik\lambda t}\, d\lambda$$

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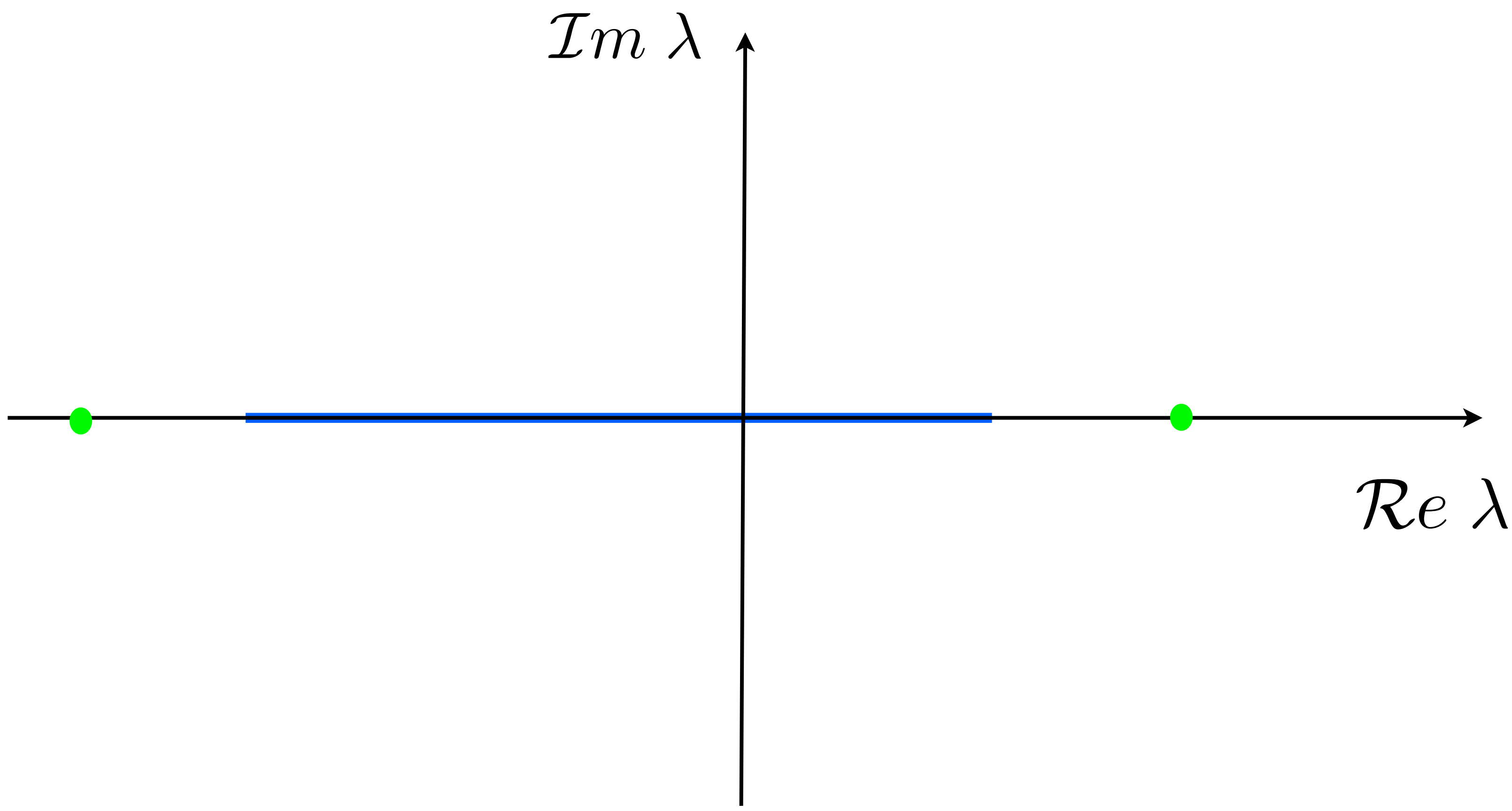
We have to use the analyticity properties of $F_M(\lambda)$.

Inverting the Laplace transform in time



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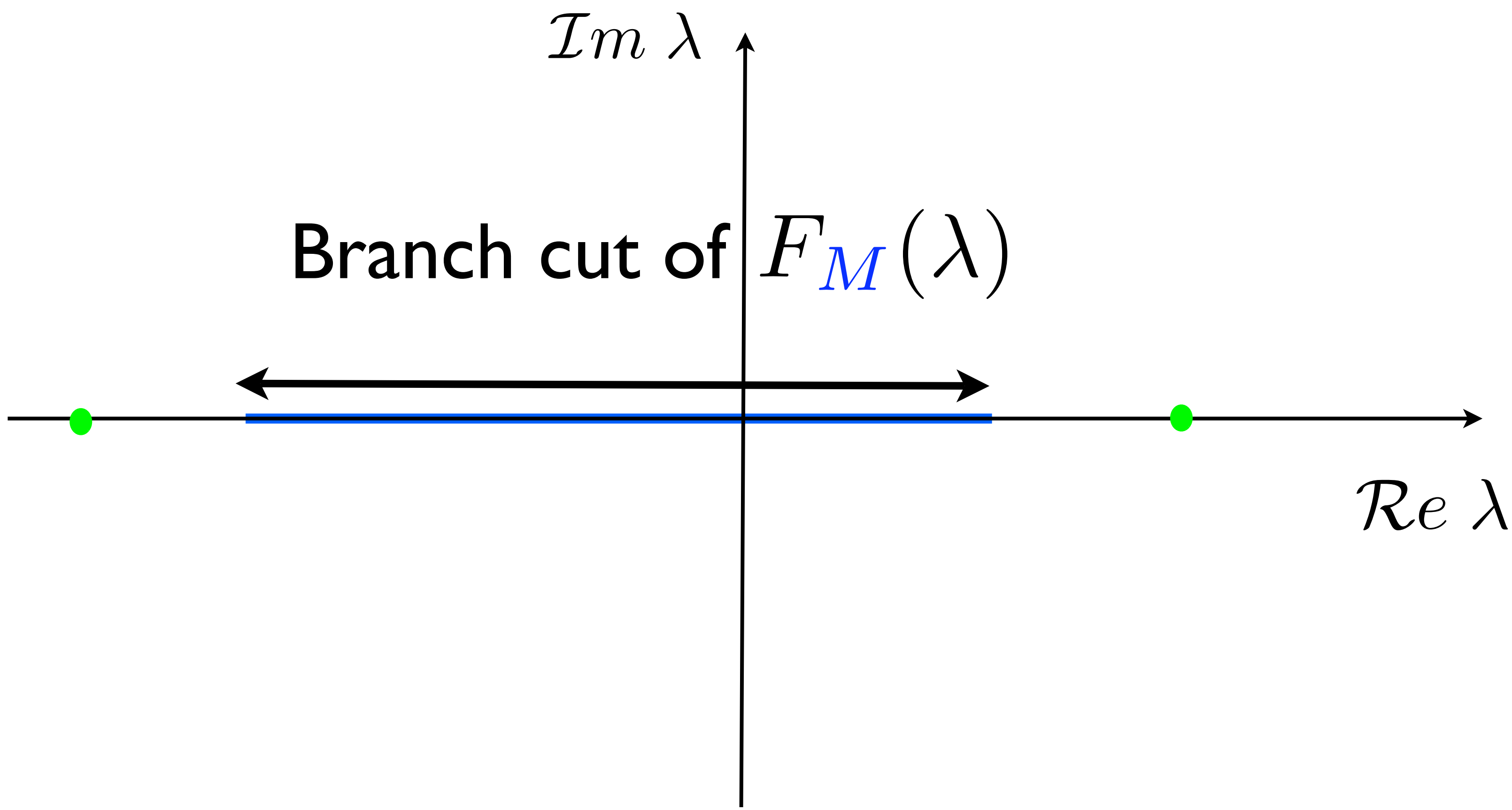

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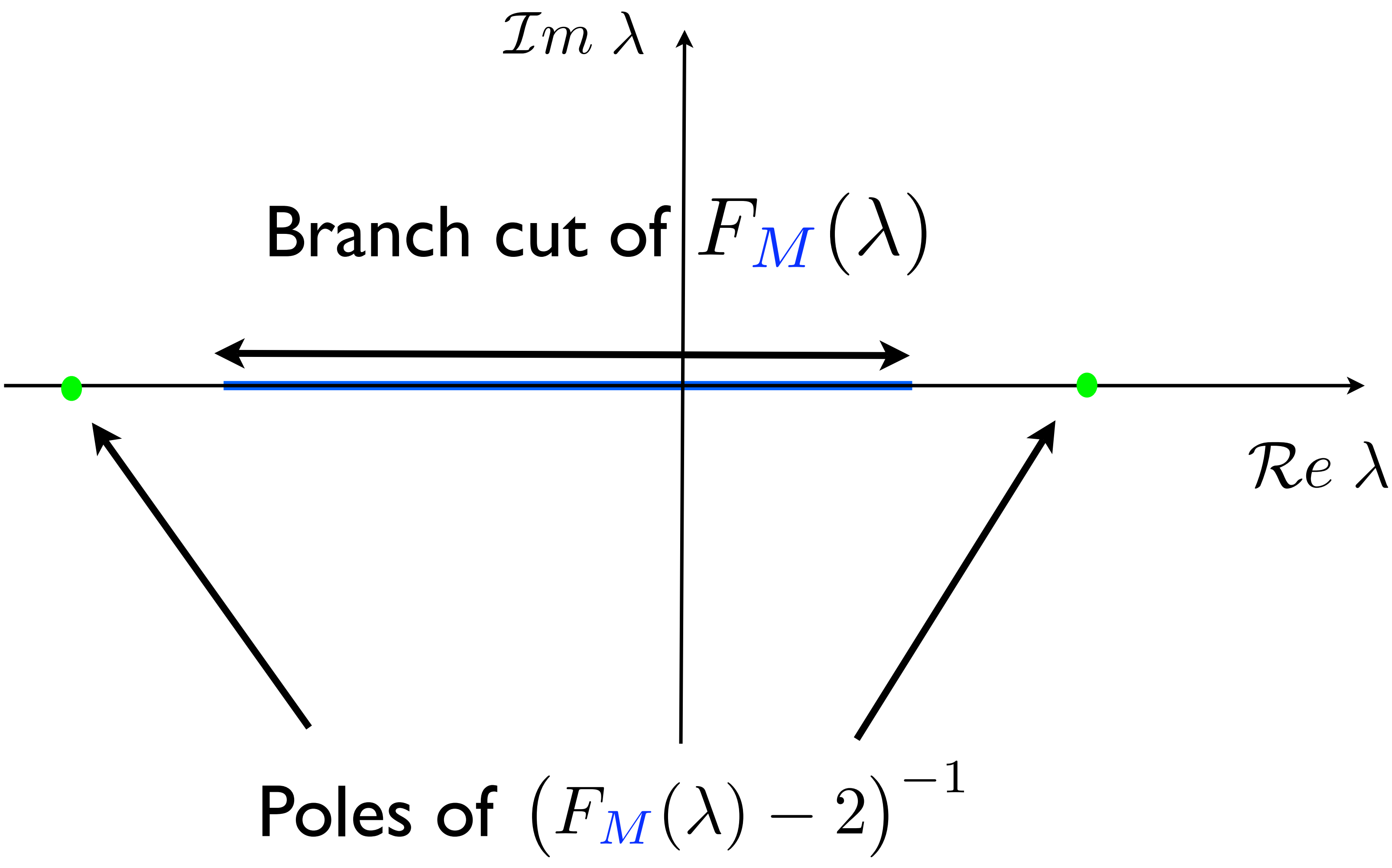

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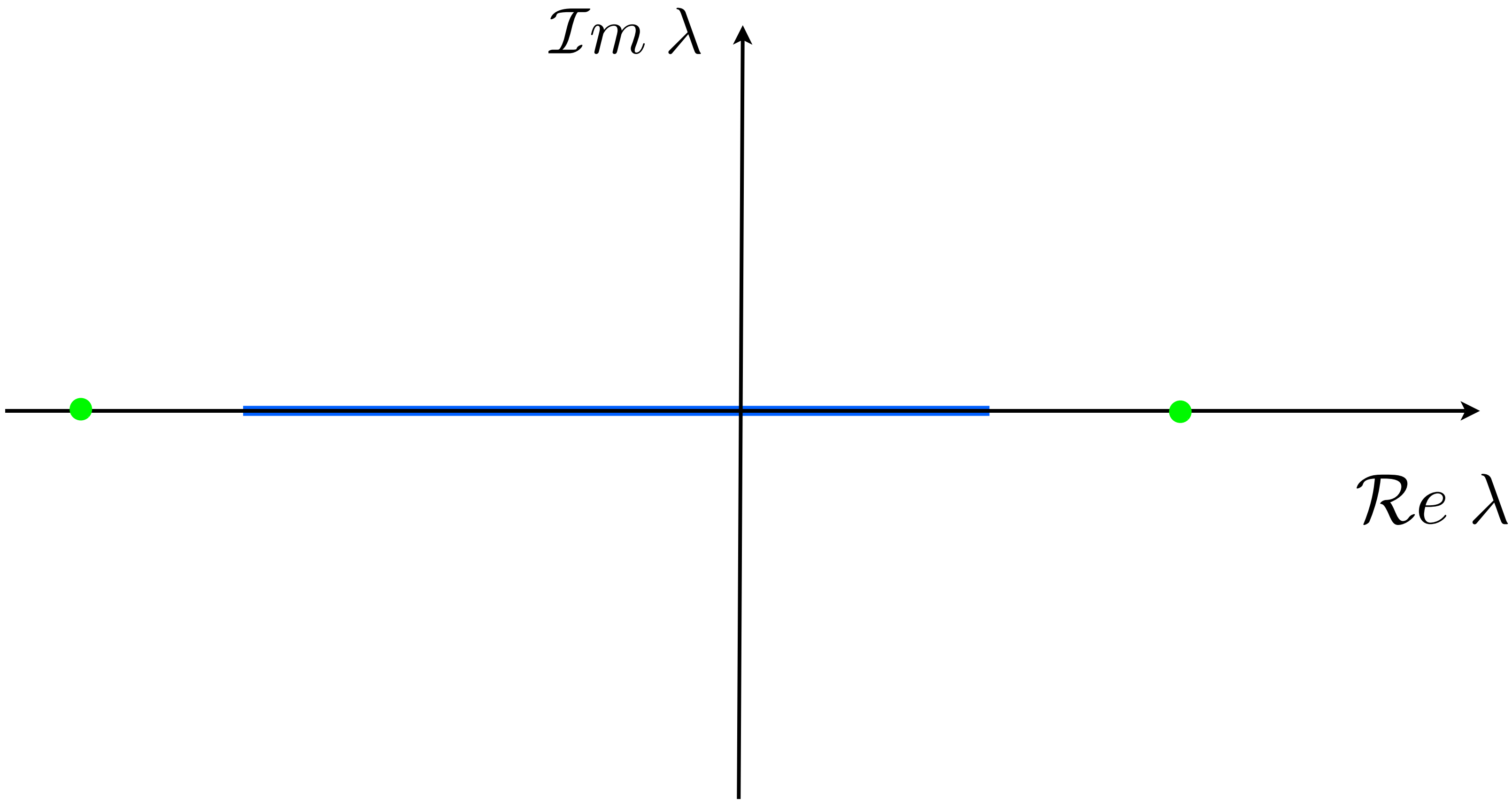

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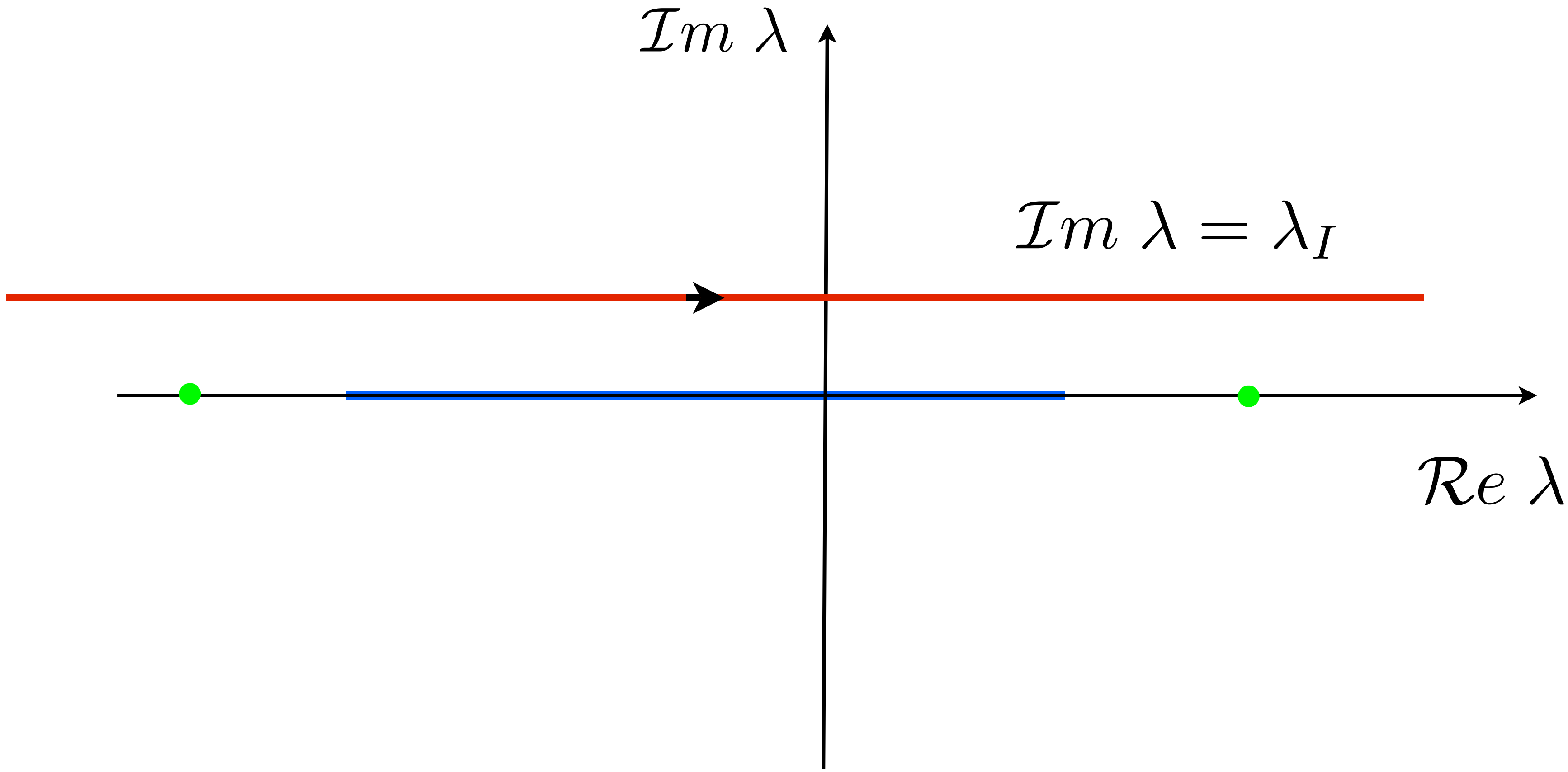

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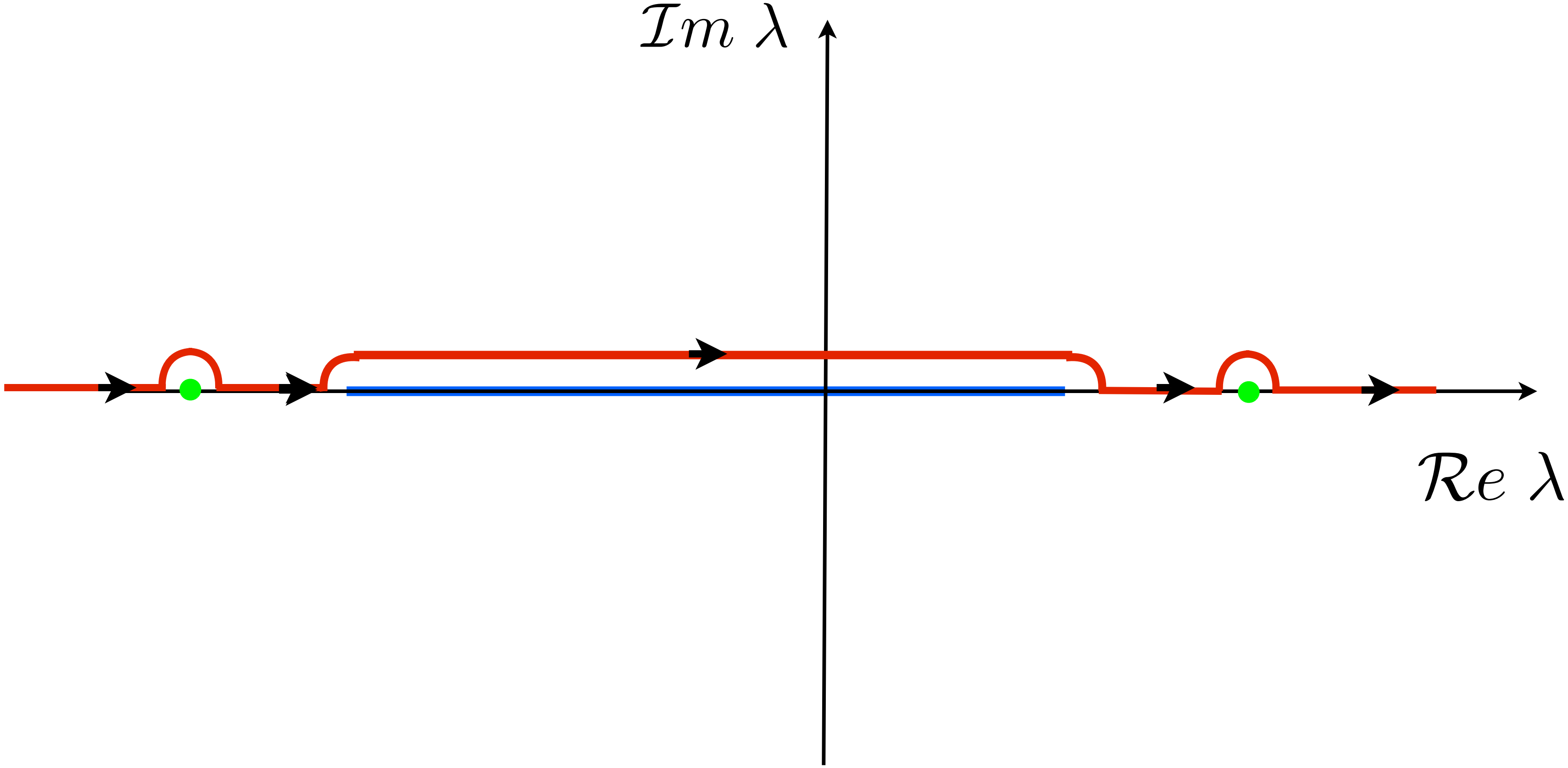

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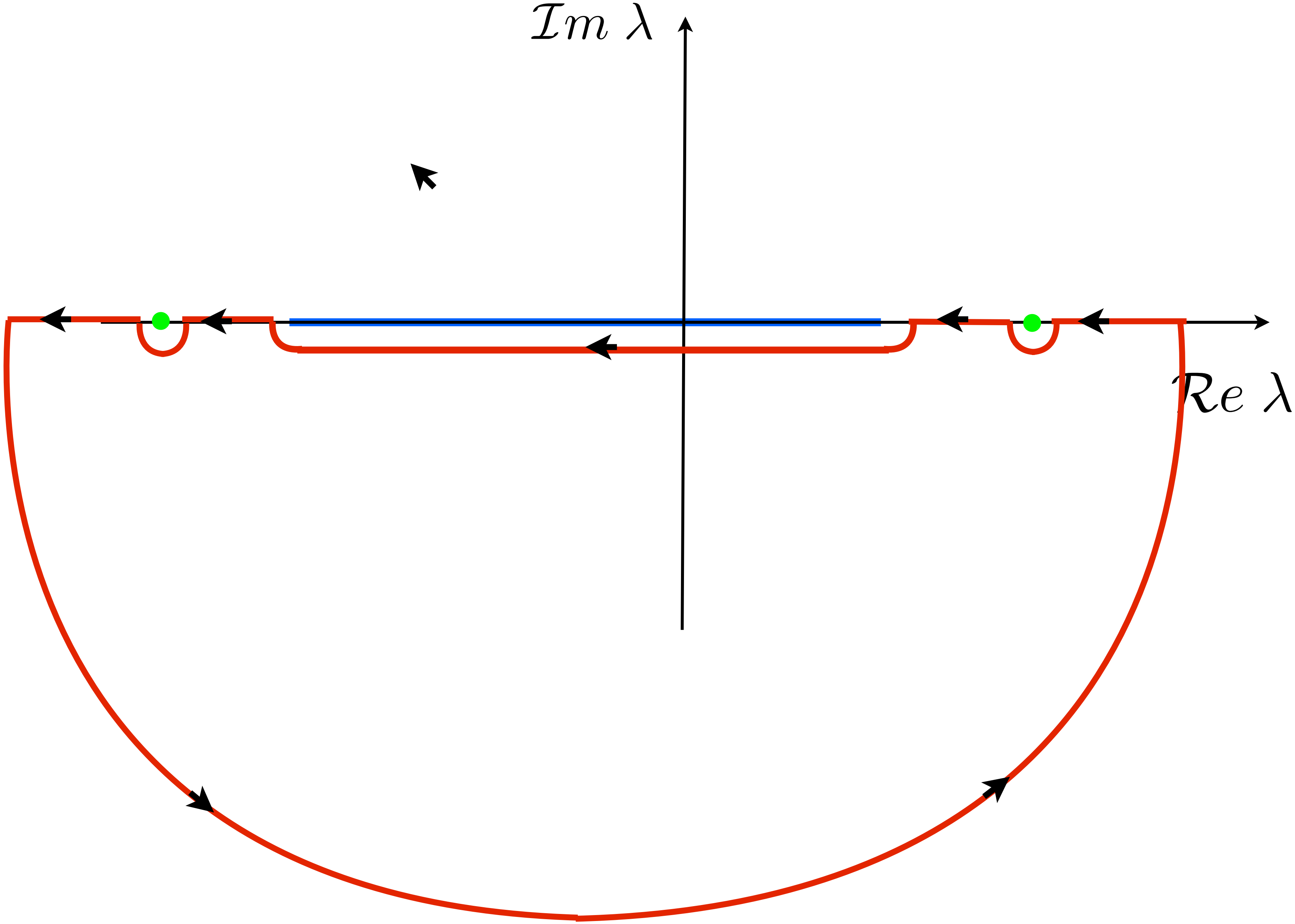

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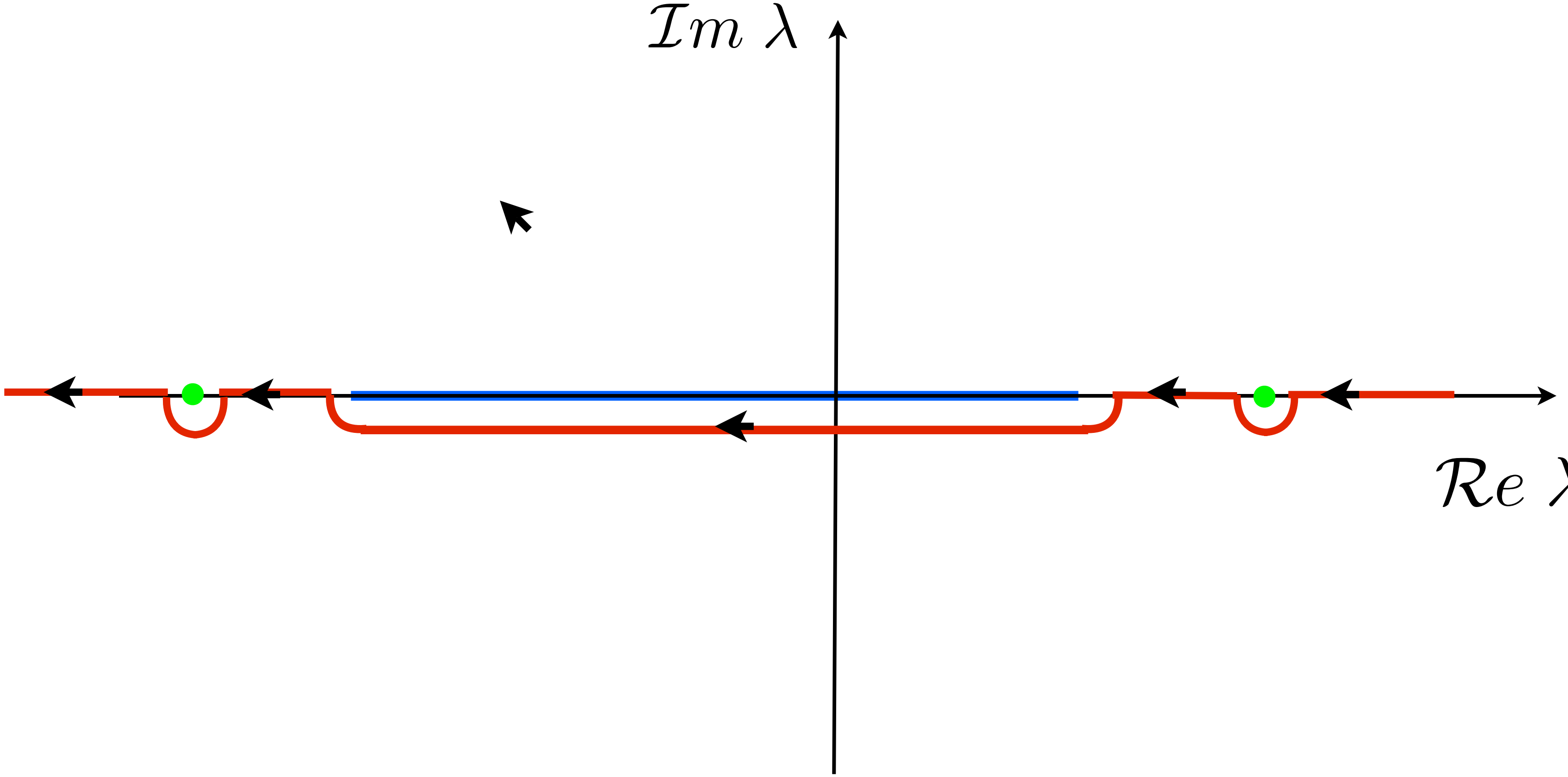

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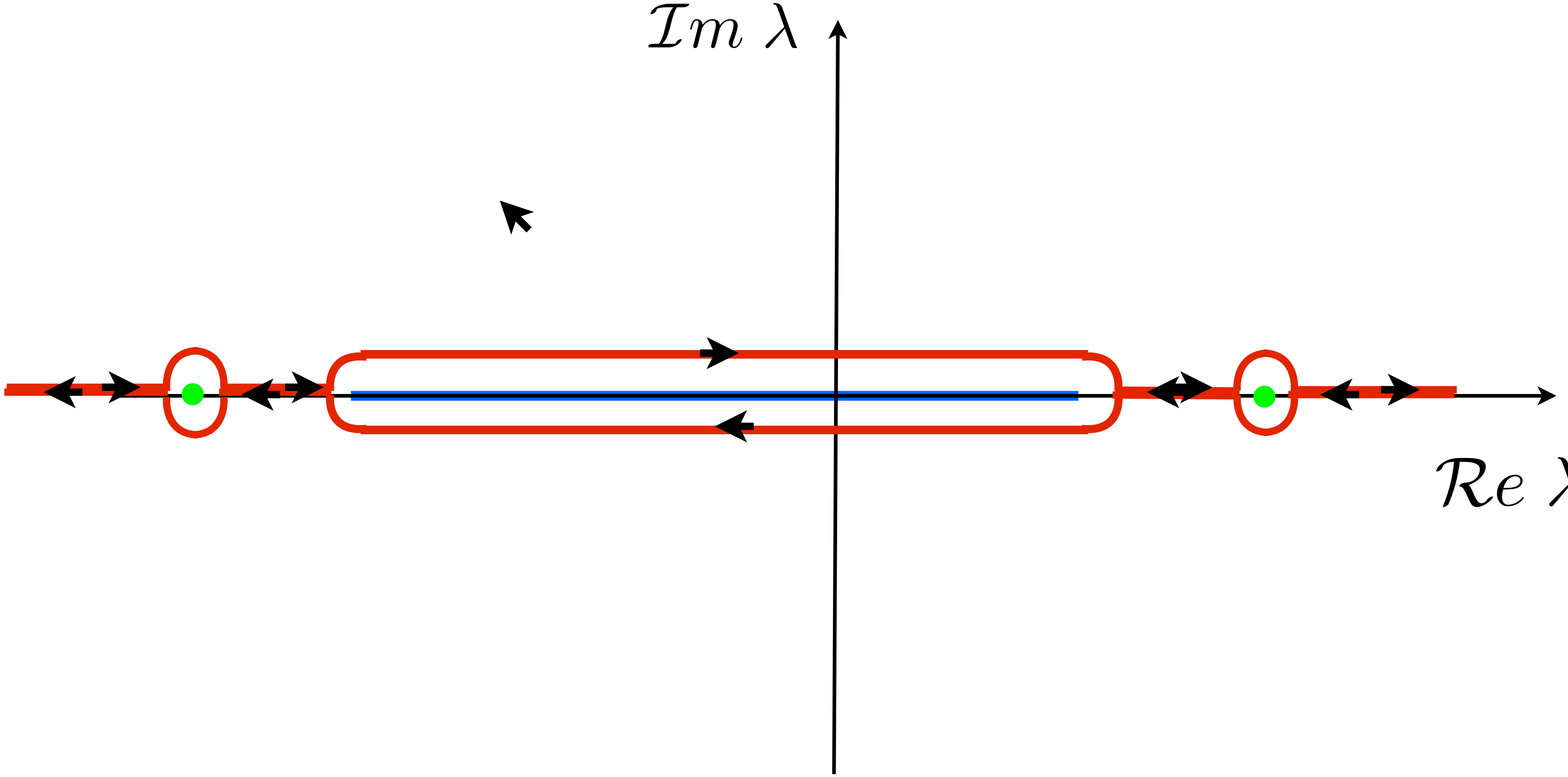

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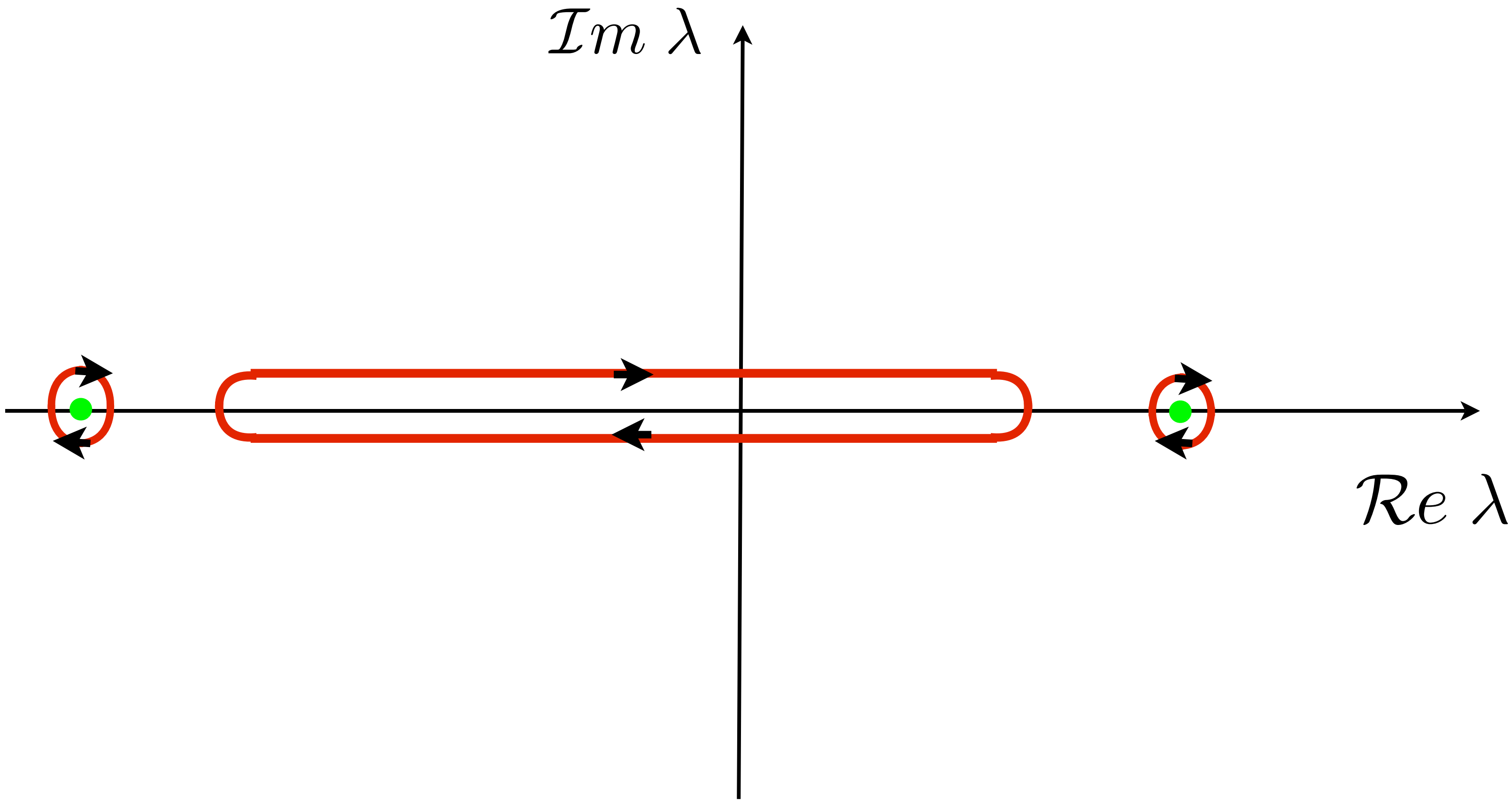

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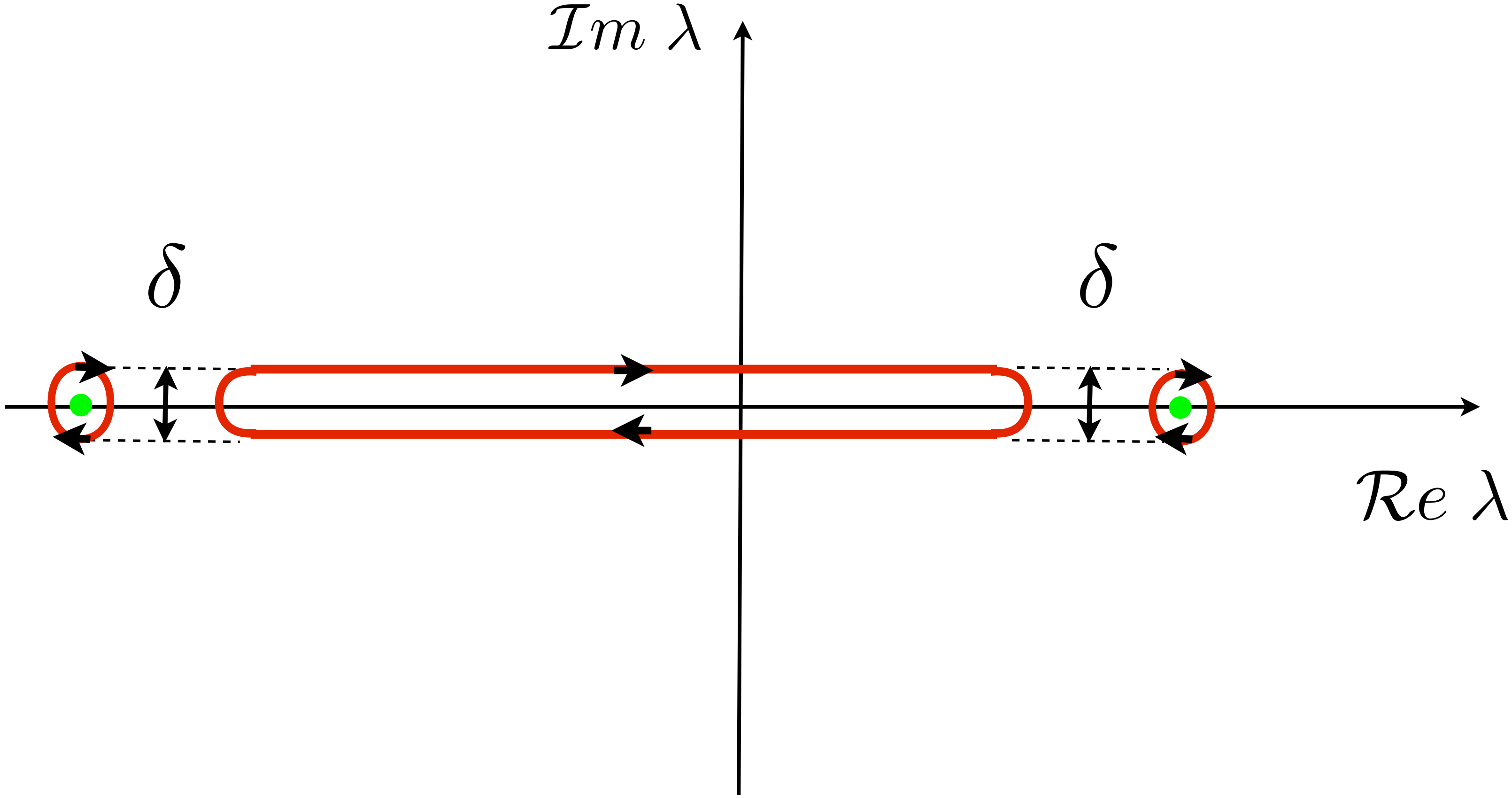

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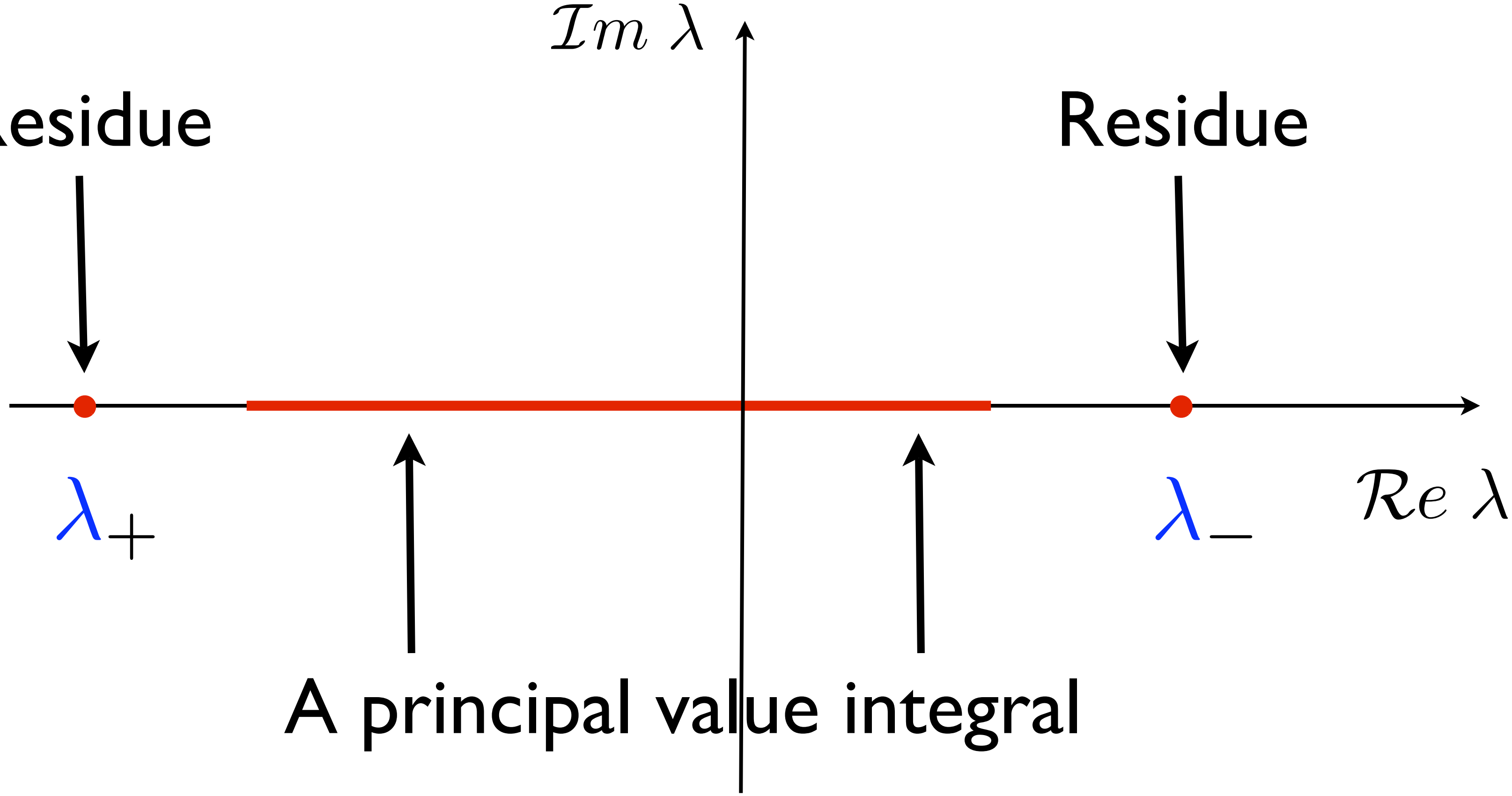

$$\left(\lim_{\delta\to 0}\int \frac{g(\lambda\pm\imath\delta)}{\lambda\pm\imath\delta}\,d\lambda = P.V.\int\frac{g_\pm(\lambda)}{\lambda}\,d\lambda \mp \imath\pi g_\pm(0)\right)$$

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U_c is a continuous superposition on λ of solutions of squared transport equations

$$U_c = \int_{M_-}^{M_+} U_{c,\lambda}\, d\lambda \qquad (\partial_t - \lambda\,\partial_x)^2\, U_{c,\lambda} = 0$$

A numerical illustration

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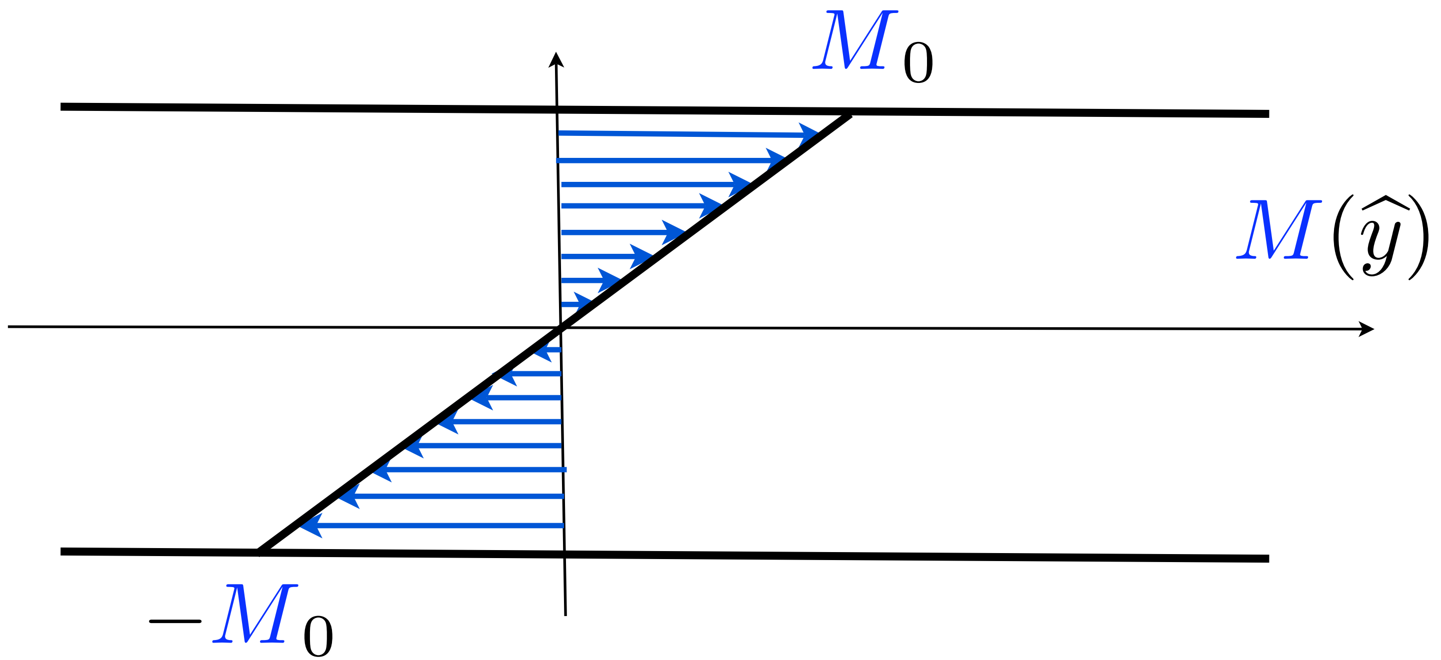

A numerical illustration

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We present a numerical result for a linear profile

$$M(y) = M_0\, y$$

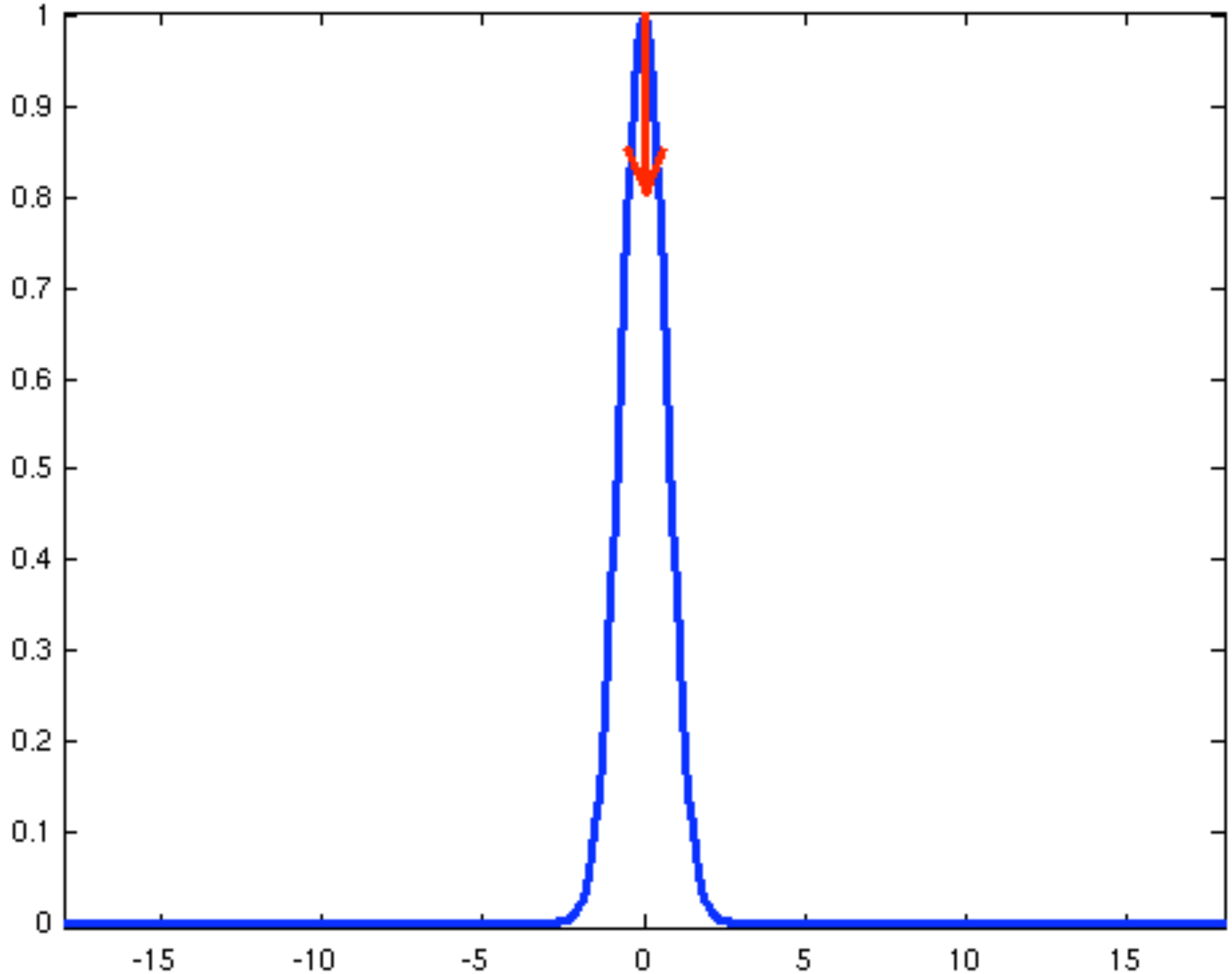
and for the following initial conditions

$$u_0(x,y) = g(x), \quad u_0(x,y) = 0.$$

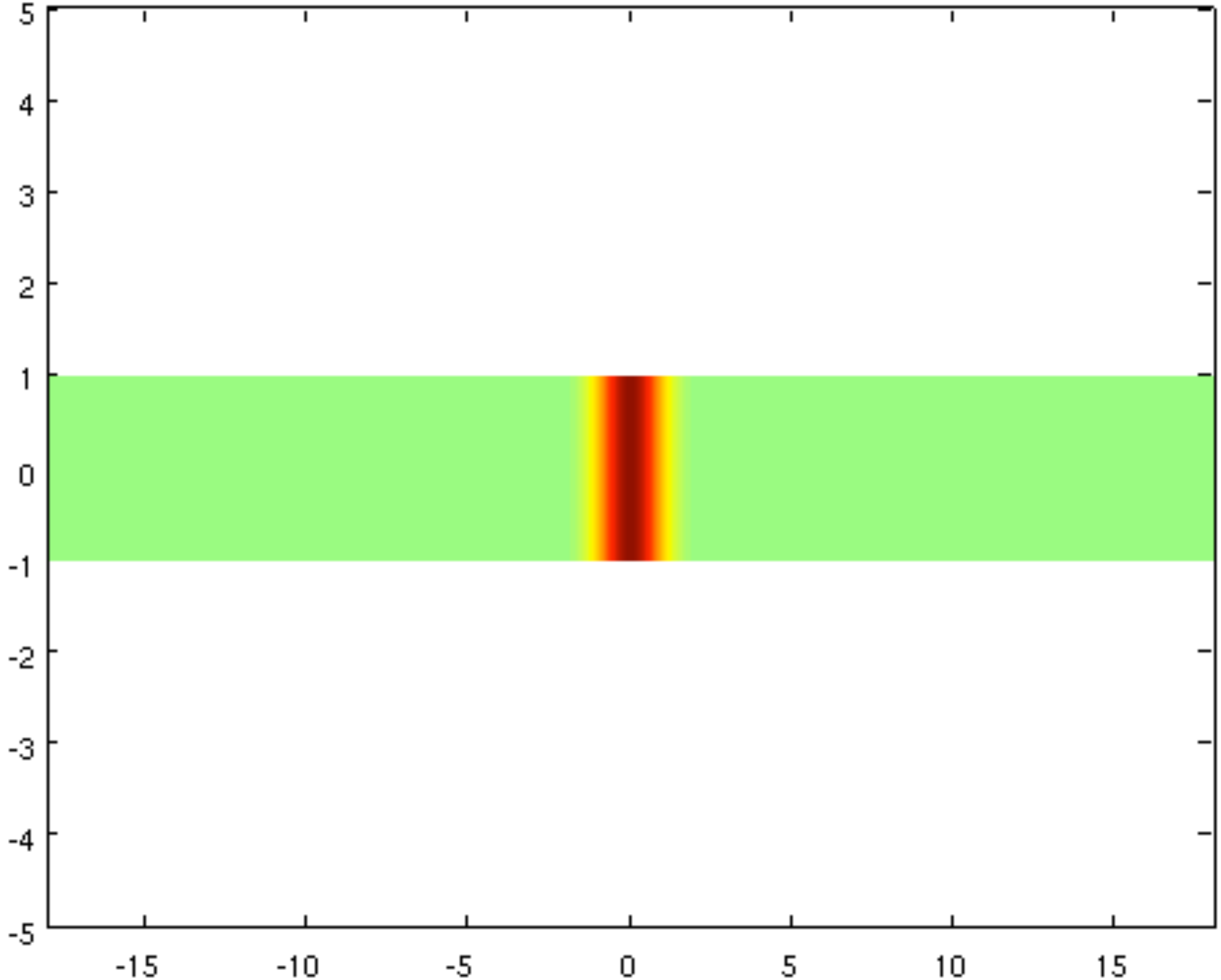
where g is a gaussian profile.

The function $U(x,t)$, $M_0 = 0.4$

The red arrows move at velocities λ_+ and $\lambda_- = -\lambda_+$



The function $u(x, y, t)$, $\quad M_0 = 0.4$



The function $U(x,t)$, $M_0 = 1$

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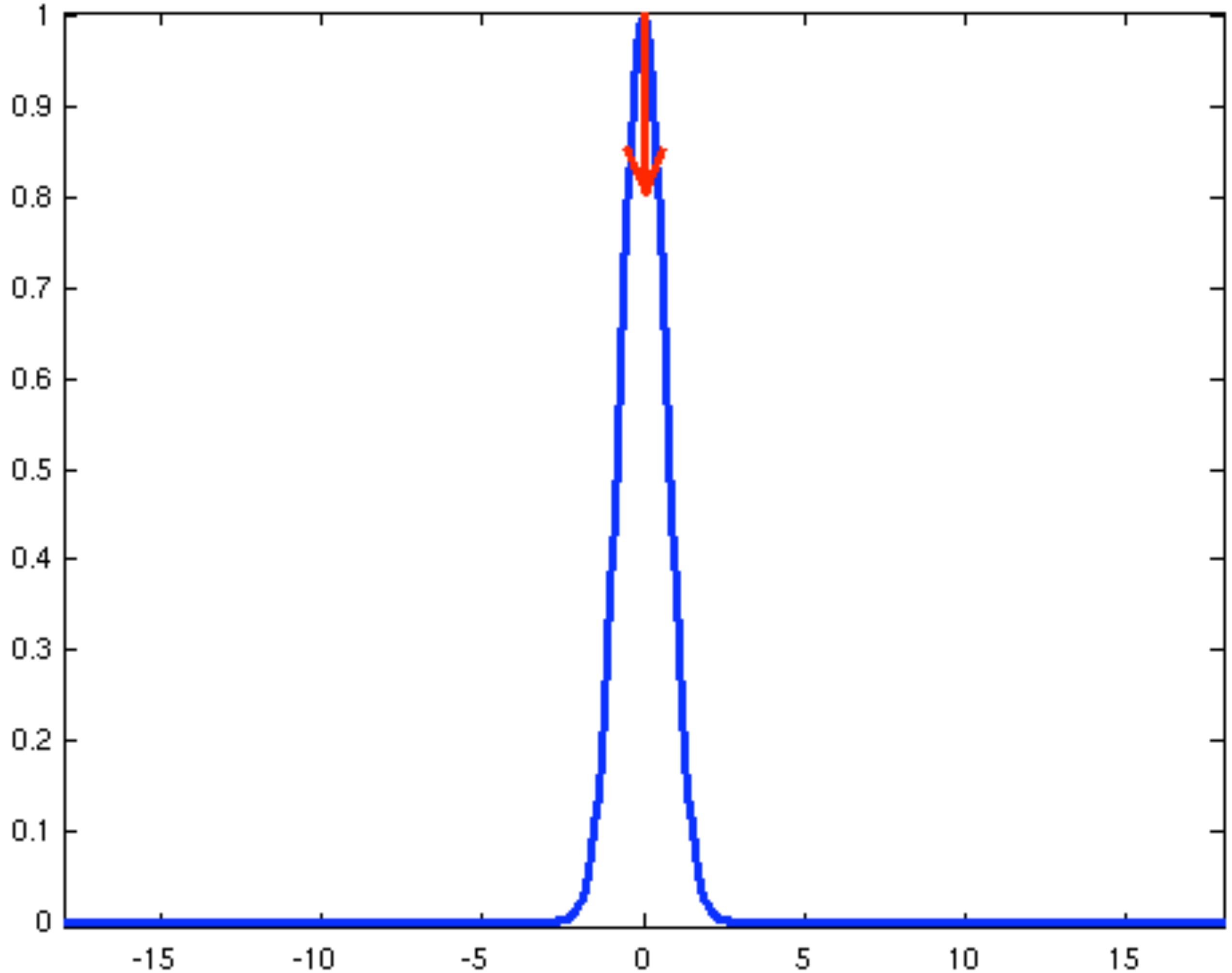

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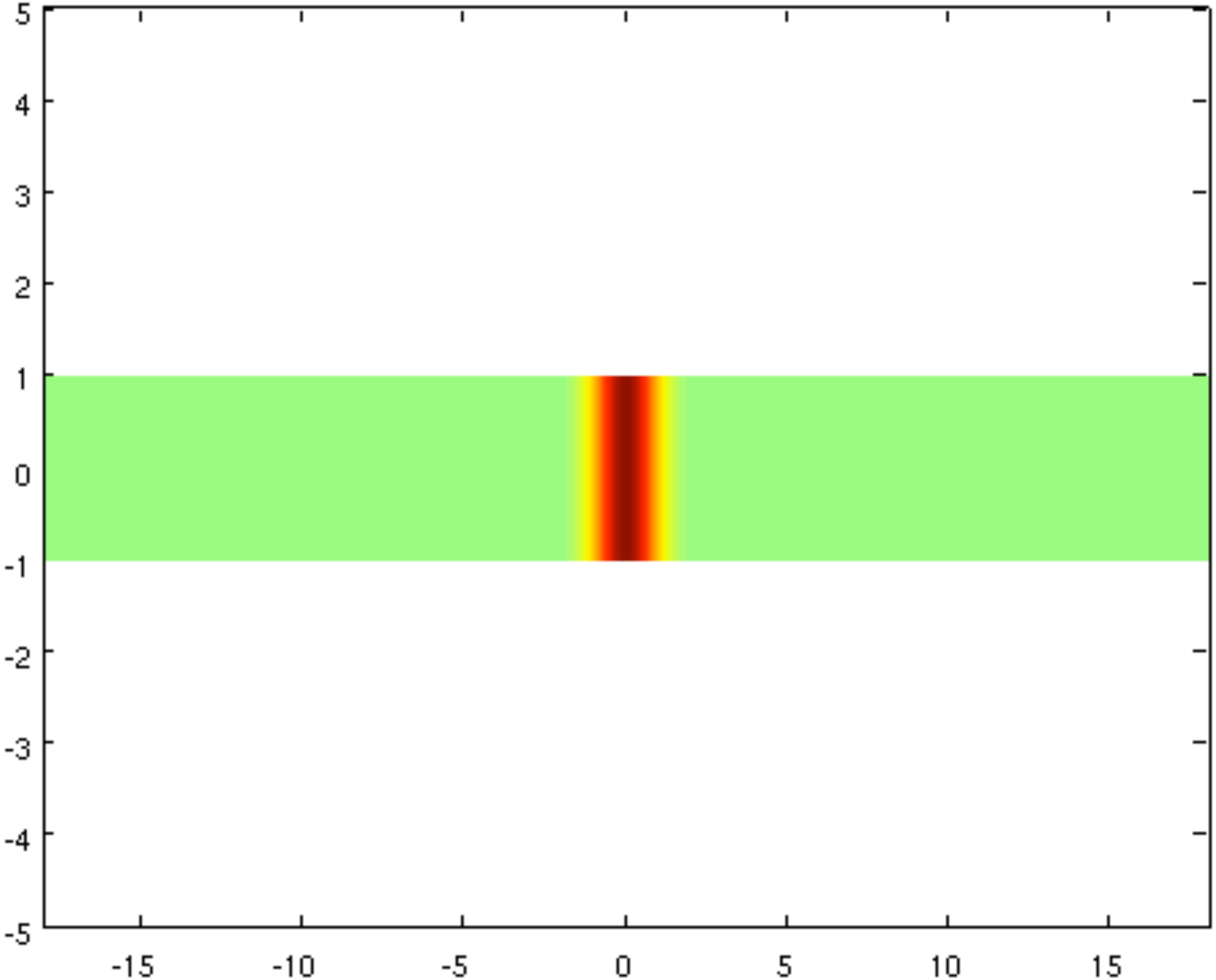

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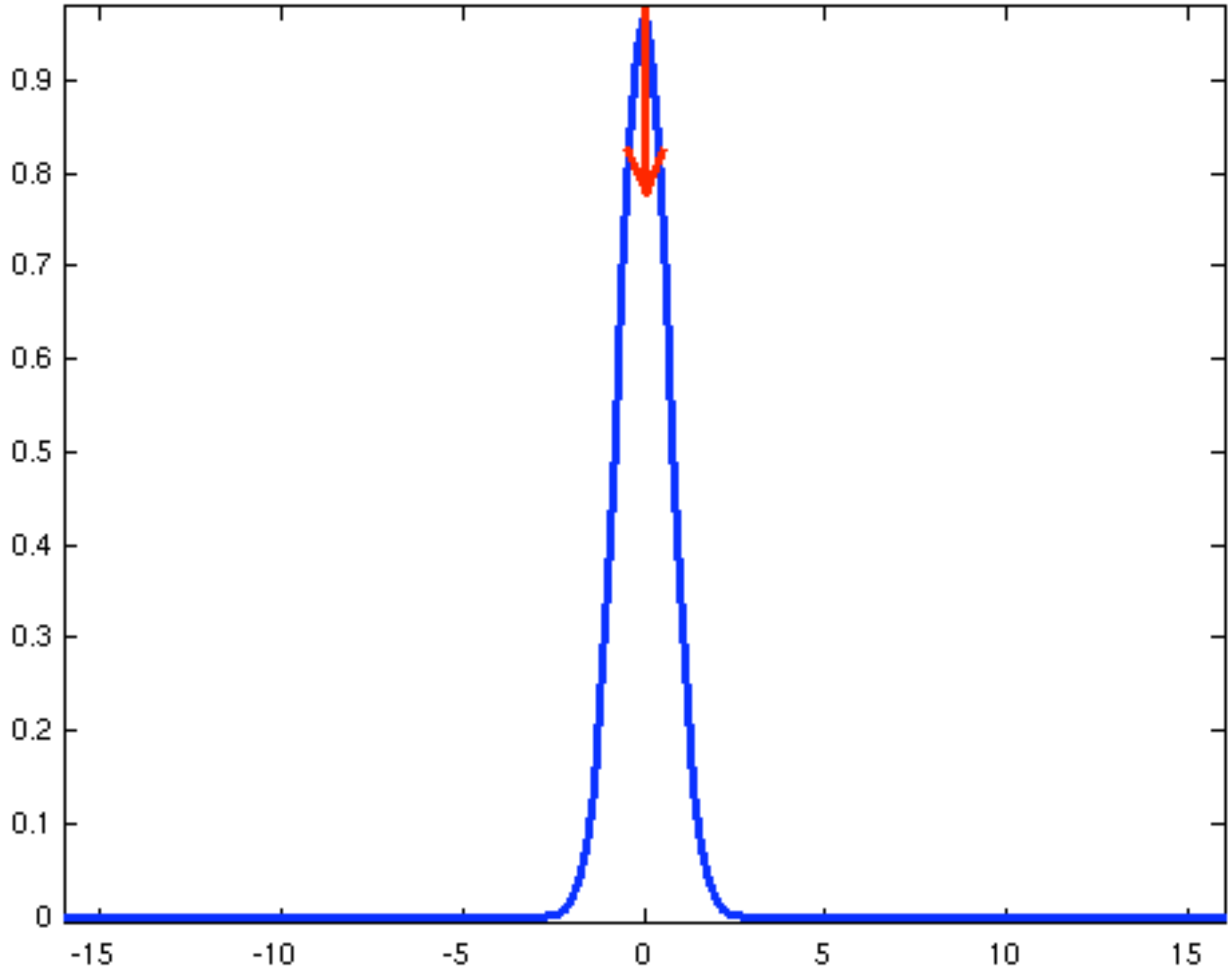

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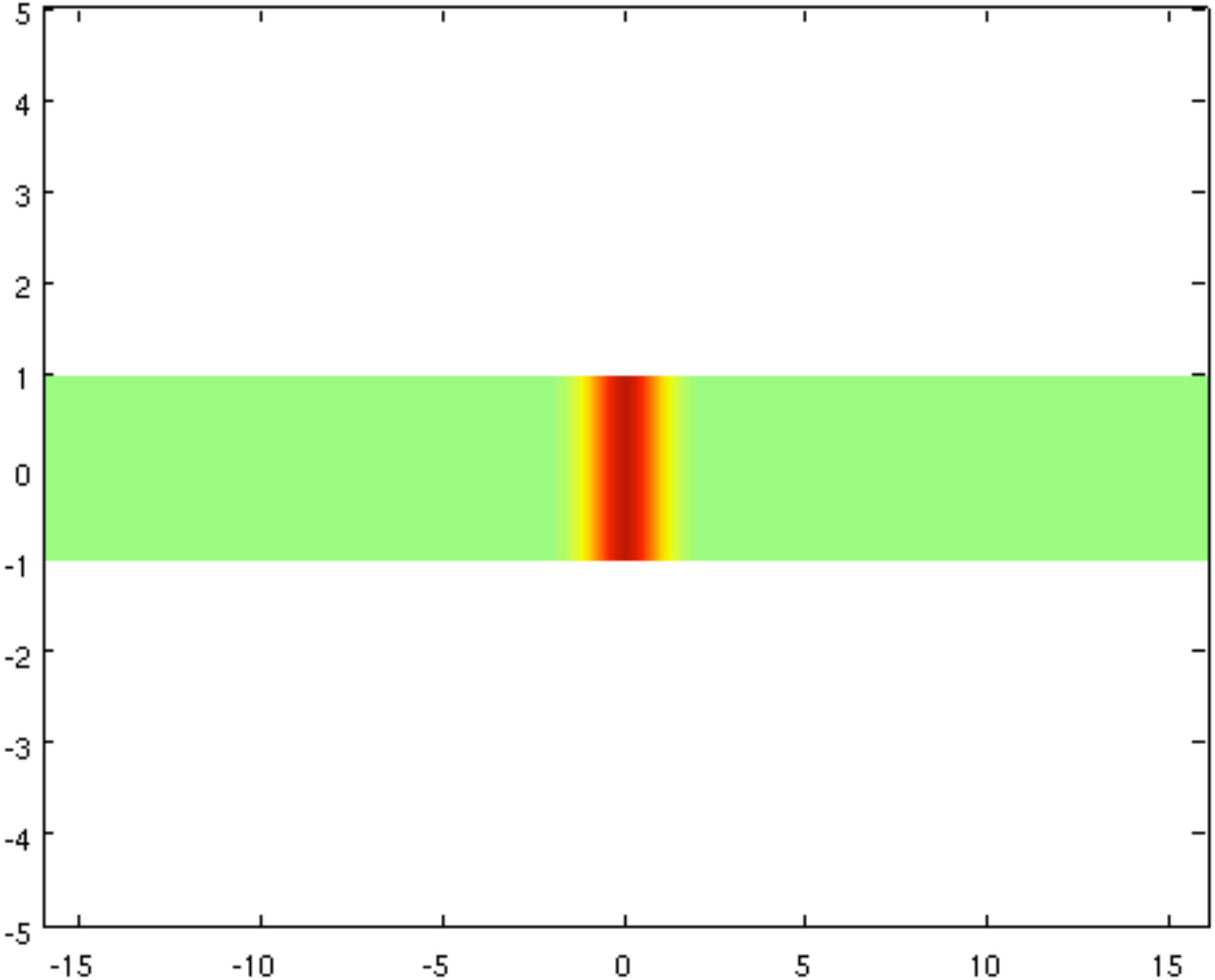

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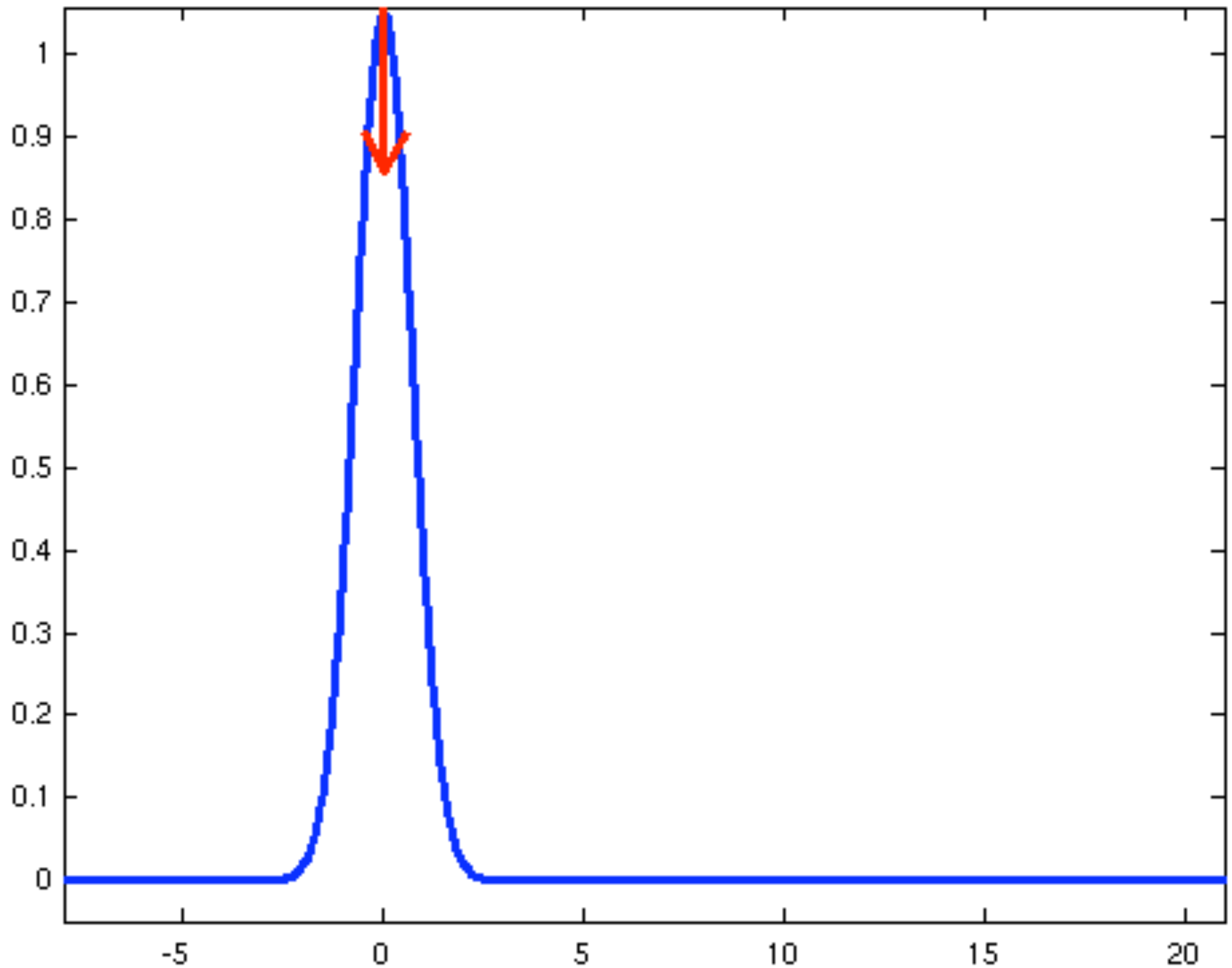

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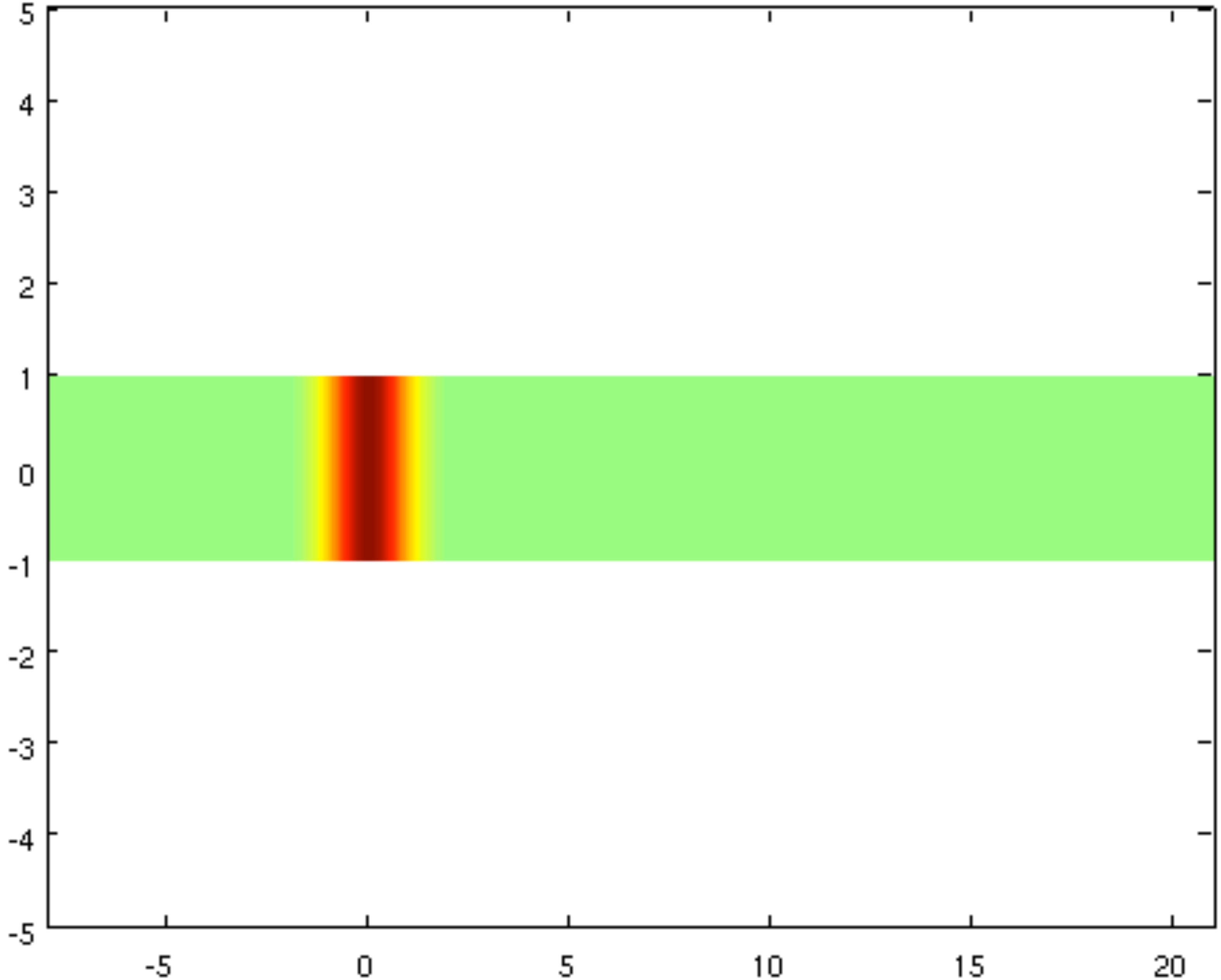
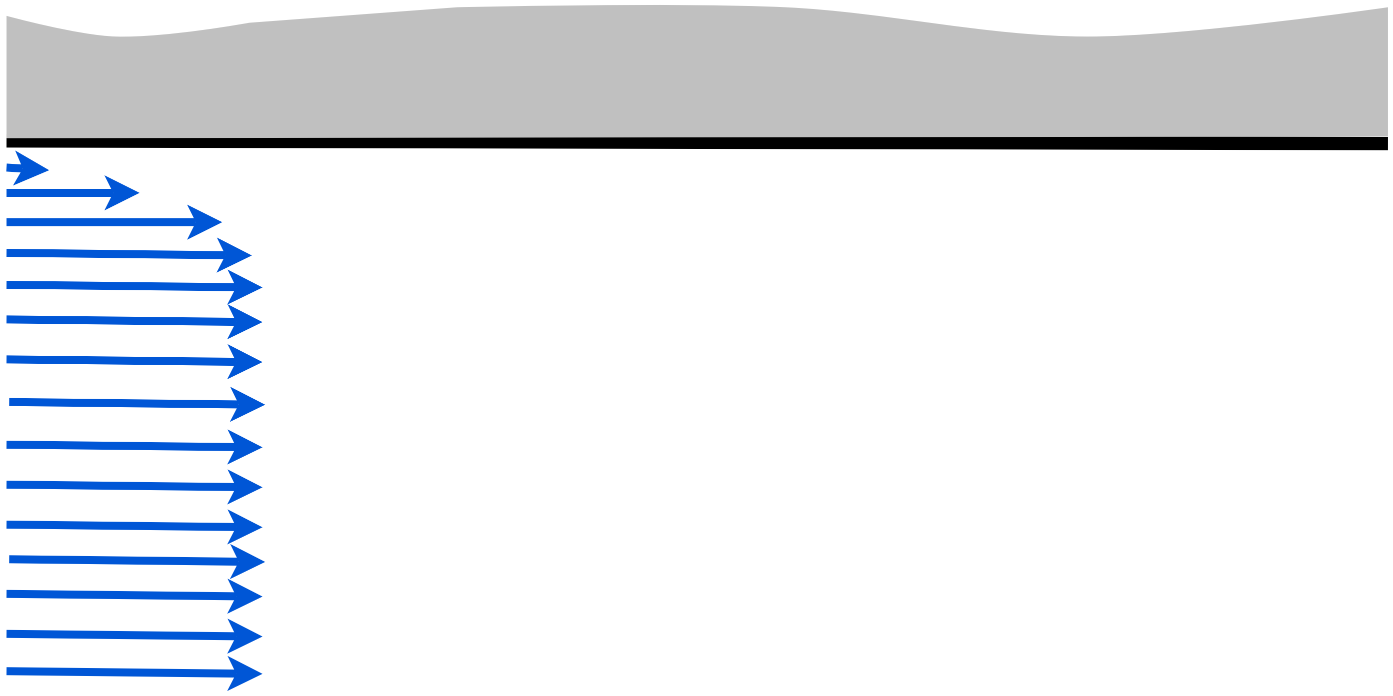


Effective boundary conditions (in progress)



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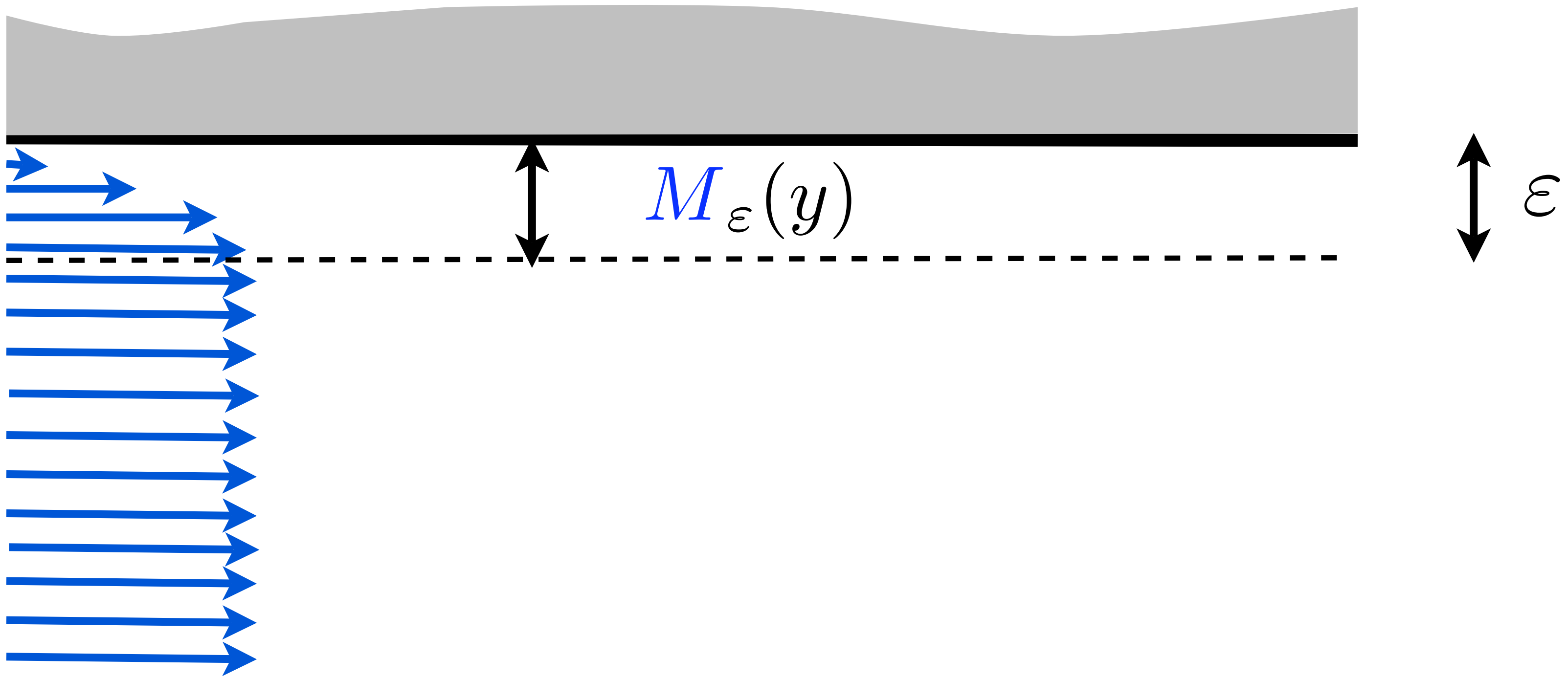

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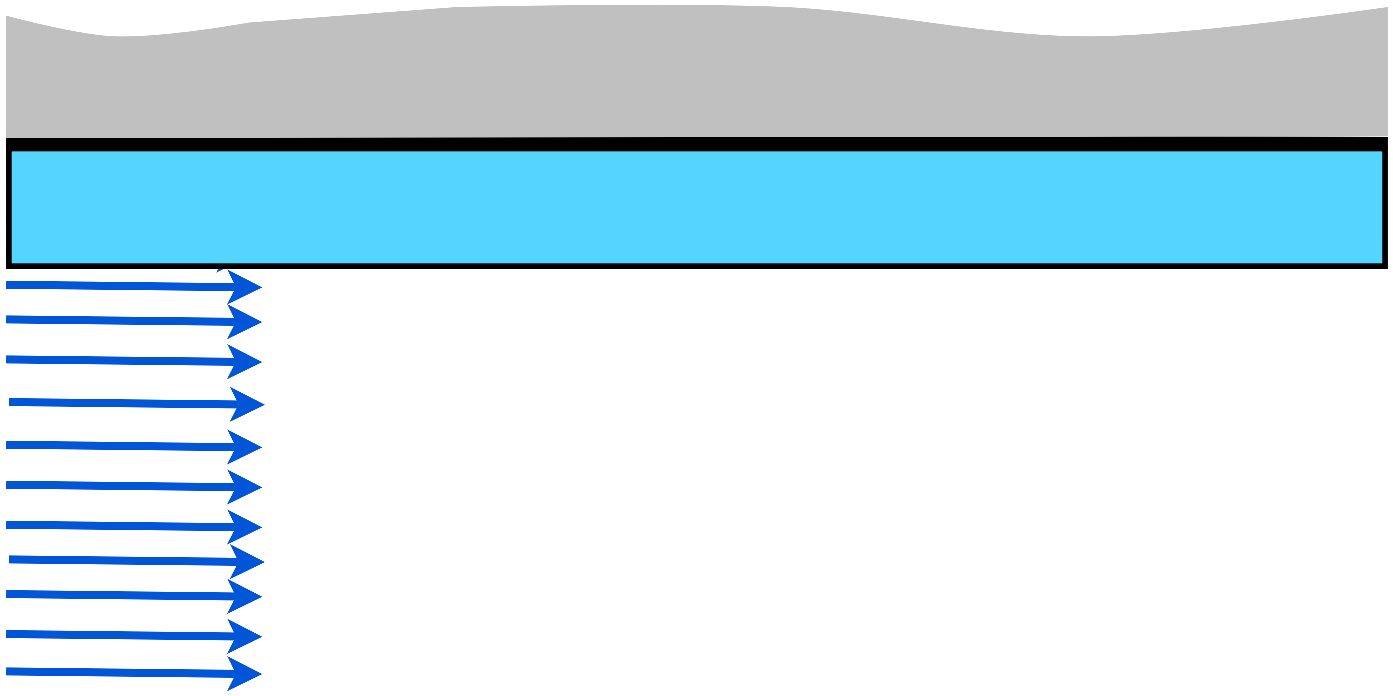

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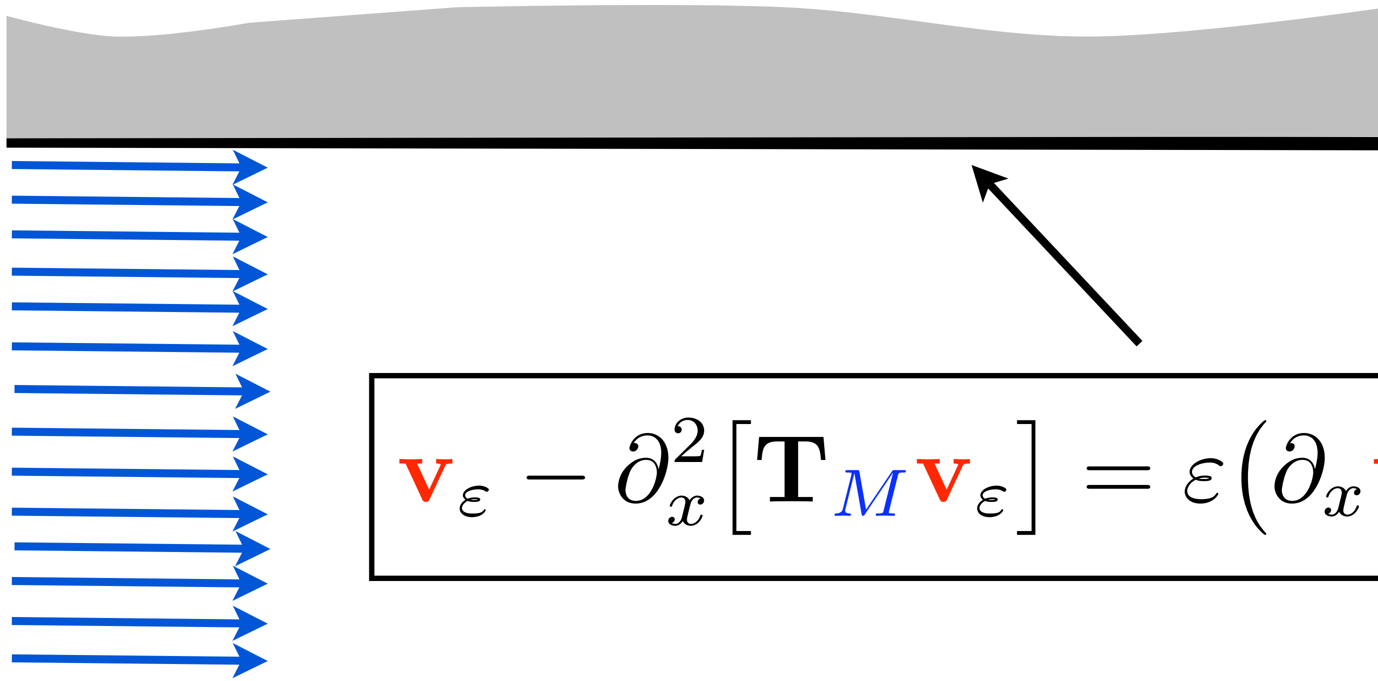

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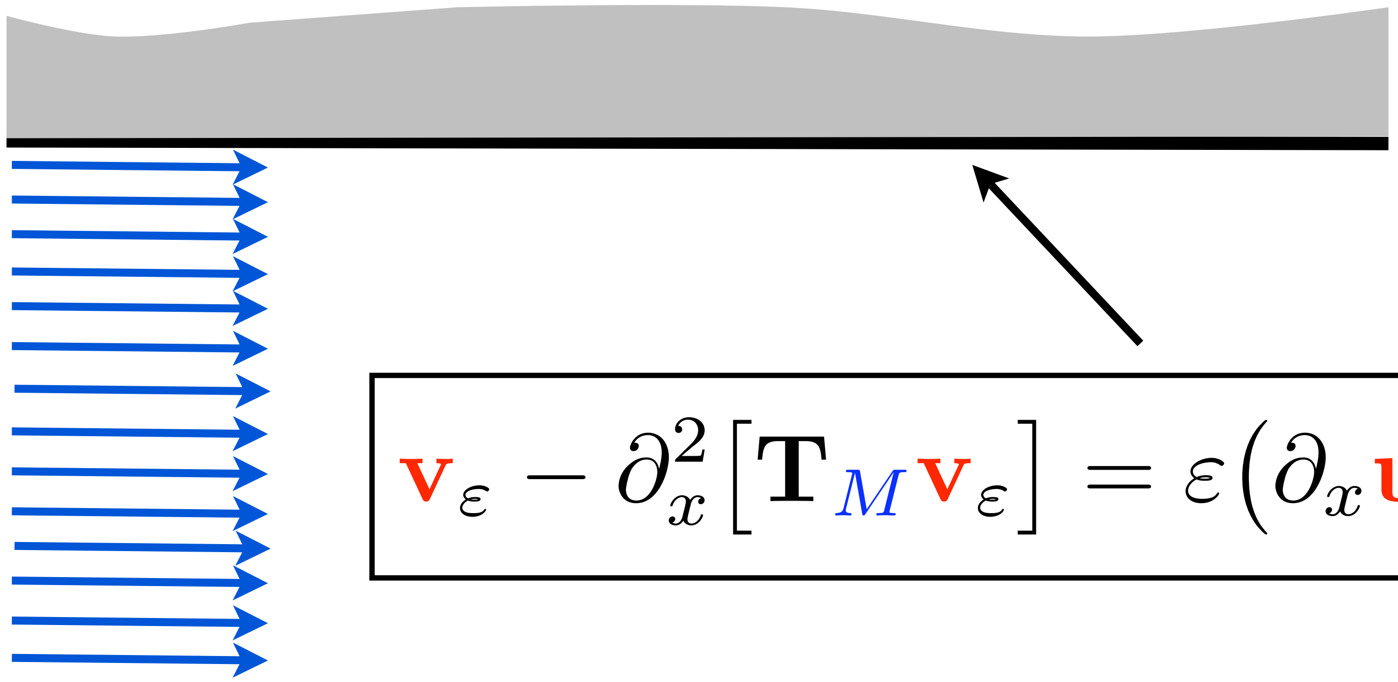

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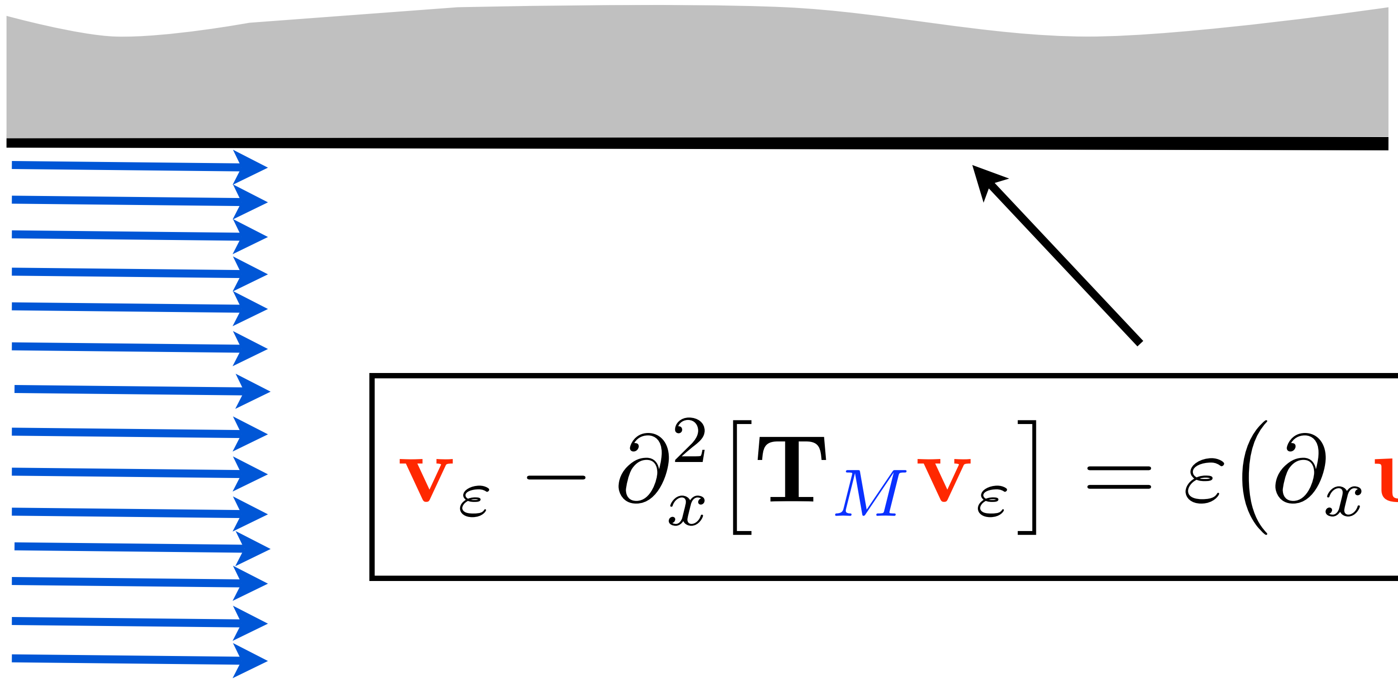

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The well-posedness of the initial boundary problem in the half-space has been proven (Kreiss method)

Effective boundary conditions (in progress)

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Questions (1)

Describe the reflection of waves

Investigate the existence of surface waves

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Questions (2)

Find an efficient numerical method