Applications of the periodic unfolding method to multi-scale problems

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December 14, 2009

Homogenization - Introduction

Let

$$A^\varepsilon(x) = (a^\varepsilon_{ij}(x))_{1\le i,j\le n} \quad \text{a.e. on } \Omega \subset \mathbb{R}^n,$$

satisfying for any $\lambda \in \mathbb{R}^n$,

$$\begin{cases} (A^\varepsilon(x)\lambda, \lambda) \ge \alpha|\lambda|^2, \\ |A^\varepsilon(x)\lambda| \le \beta|\lambda|, \end{cases}$$

Set

$$\mathcal{A}_\varepsilon = -\text{div }(A^\varepsilon \nabla) = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\big(a^\varepsilon_{ij}\frac{\partial}{\partial x_j}\big).$$

Consider the equation

$$\mathcal{A}_\varepsilon u^\varepsilon = f,$$

with, for instance, a Dirichlet condition on $\partial\Omega$.

One has to solve

$$\begin{cases} -\text{div}\,(A^{\varepsilon}\nabla u^{\varepsilon}) = f & \text{in } \Omega \\ u^{\varepsilon} = 0 & \text{on } \partial\Omega. \end{cases}$$

with $f \in H^{-1}(\Omega)$. By the Lax-Milgram theorem there exists a unique $u^{\varepsilon} \in H_0^1(\Omega)$ such that

$$\int_{\Omega} A^{\varepsilon}\nabla u^{\varepsilon}\,\nabla v\,dx = \langle f, v\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}, \quad \forall v \in H_0^1(\Omega).$$

Moreover, since

$$\|u^{\varepsilon}\|_{H_0^1(\Omega)} \leq \frac{1}{\alpha}\|f\|_{H^{-1}(\Omega)},$$

there exists $u^0 \in H_0^1(\Omega)$ such that

$$u^{\varepsilon'} \rightharpoonup u^0 \quad \text{weakly in } H_0^1(\Omega).$$

Set

$$\xi^\varepsilon = (\xi_1^\varepsilon, \dots, \xi_n^\varepsilon) = \Big(\sum_{j=1}^n a_{1j}^\varepsilon \frac{\partial u^\varepsilon}{\partial x_j}, \dots, \sum_{j=1}^n a_{nj}^\varepsilon \frac{\partial u^\varepsilon}{\partial x_j}\Big) = A^\varepsilon \nabla u^\varepsilon,$$

which satisfies

$$\int_\Omega \xi^\varepsilon \, \nabla v \, dx \;=\; \langle f, v\rangle_{H^{-1}(\Omega), H_0^1(\Omega)}, \quad \forall v \in H_0^1(\Omega).$$

There exists $\xi^0 \in L^2(\Omega)$, such that (up to a subsequence)

$$\xi^{\varepsilon'} \rightharpoonup \xi^0 \quad \text{weakly in } (L^2(\Omega))^n.$$

Moreover,

$$\int_\Omega \xi^0 \, \nabla v \, dx \;=\; \langle f, v\rangle_{H^{-1}(\Omega), H_0^1(\Omega)}, \quad \forall v \in H_0^1(\Omega),$$

i.e.

$$-\text{div}\, \xi^0 = f \quad \text{in } \Omega.$$

- Question : the relation between ξ^0 and u^0 ?

Several methods give the answer :

- the multiple-scale method,
- the Tartar's oscillating test functions method,
- the two-scale convergence,
- the periodic unfolding method.

Multiple-scale method

Suppose now that

$$A^\varepsilon(x) = A\Big(\frac{x}{\varepsilon}\Big), \quad A(y) = (a_{ij}(y))_{1\le i,j\le n},\ a_{ij}\ Y - \text{periodic}.$$

One looks for

$$u^\varepsilon(x) = u_0\Big(x, \frac{x}{\varepsilon}\Big) + \varepsilon u_1\Big(x, \frac{x}{\varepsilon}\Big) + \varepsilon^2 u_2\Big(x, \frac{x}{\varepsilon}\Big) + \cdots$$

with $u_j(x, y)$ for $j = 1, 2, \ldots$ such that

$$\begin{cases} u_j(x, y) \text{ is defined for } x \in \Omega \text{ and } y \in Y \\ u_j(\cdot, y) \text{ is } Y - \text{periodic.} \end{cases}$$

Tartar's oscillating test function method

Let $\varphi \in \mathcal{D}(\Omega)$ et z_ε a sequence of functions such that $z_\varepsilon \varphi \in H^1_0(\Omega)$. With z_ε as test fuctions,

$$\int_\Omega \varphi\, a_{ij}\Big(\frac{x}{\varepsilon}\Big)\frac{\partial u^\varepsilon}{\partial x_j}\,\frac{\partial z_\varepsilon}{\partial x_i} + \int_\Omega z_\varepsilon\, a_{ij}\Big(\frac{x}{\varepsilon}\Big)\frac{\partial u^\varepsilon}{\partial x_j}\,\frac{\partial \varphi}{\partial x_i} = \int_\Omega f\, z_\varepsilon \varphi\, dx.$$

Idea : construction of z_ε as the solution of the adjoint system $\Longrightarrow$

$$\int_\Omega \varphi\, a^*_{ij}\Big(\frac{x}{\varepsilon}\Big)\frac{\partial z_\varepsilon}{\partial x_j}\,\frac{\partial u^\varepsilon}{\partial x_i} + \int_\Omega u^\varepsilon\, a^*_{ij}\Big(\frac{x}{\varepsilon}\Big)\frac{\partial z_\varepsilon}{\partial x_j}\,\frac{\partial \varphi}{\partial x_i} = 0.$$

$\Longrightarrow$ The bad terms cancel and one can pass to the limit in all the others.

Two-scale method (G. Nguetseng et G. Allaire

Let $\{v^\varepsilon\}$ a sequence of functions in $L^2(\Omega)$. One says that $\{v^\varepsilon\}$ is two-scale convergent to $v_0 = v_0(x,y)$ with $v_0 \in L^2(\Omega \times Y)$ if for any $\psi = \psi(x,y) \in \mathcal{D}(\Omega; C^\infty_{per}(Y))$, one has

$$\lim_{\varepsilon\to 0} \int_\Omega v^\varepsilon(x)\,\psi(x, \frac{x}{\varepsilon})dx = \frac{1}{|Y|}\int_\Omega\int_Y v_0(x,y)\,\psi(x,y)\,dydx.$$

Theorem. Let $\{v^\varepsilon\}$ be a bounded sequence in $L^2(\Omega)$. There exists $\{v^{\varepsilon'}\}$ and a function $v_0 \in L^2(\Omega \times Y)$ such that $\{v^{\varepsilon'}\}$ two-scale converges to v_0.

Periodic unfolding

Joint works with Alain Damlamian, Patrizia Donato, Georges Griso and Rachad Zaki.

Main topics

I. The periodic unfolding method in fixed domains

II. Multi-scale domains

III. Unfolding in perforated domains

IV. Application to multi-structures…

I. The periodic unfolding method in fixed domains

The periodic unfolding method is a "fixed domain" method, well suited to treat periodic homogenization problems. The basic idea is that if the proper scaling is used, oscillatory behaviour can be turned into weak or even strong convergence, at the price of an increase in the dimension of the problem, but with significant simplifications in the proofs and explicit formulas. It also gives a most elementary proof for the results of the theory of two-scale convergence due to G. Nguetseng and G. Allaire.

Originally, the periodic unfolding method was applied for linear problems in the standard periodic case of homogenization with applications to multi-scale problems. Then it was extended to monotone operators, perforated domains, nonlinear integral energies, ...

★ References

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D. C, Alain Damlamian and Georges Griso, The periodic unfolding method in homogenization, SIAM J. of Math. Anal. Vol. 40, 4 (2008), 1585–1620.

D. C, P. Donato and R. Zaki, The periodic unfolding method in perforated domains, Portugaliae Mathematica, 63, 4 (2006), 467–496.
D. C, P. Donato and R. Zaki, Asymptotic behavior of elliptic problems in perforated domains with nonlinear boundary conditions, Asymptotic Analysis, 53, 4 (2007), 209–235.

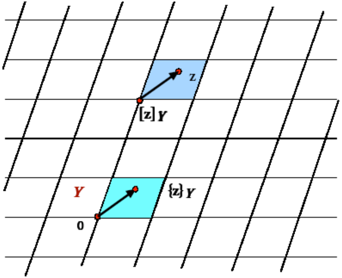
Plan

1. The periodic unfolding operator $\mathcal{T}_e$

$\Omega \subset \mathbb{R}^n$, Y a reference cell (ex. $]0,1[^n$), or more generally a set having the paving property with respect to a basis $(b_1, \cdots, b_n)$.
For $z \in \mathbb{R}^n$, $[z]_Y$ denotes the unique integer combination $\sum_{j=1}^n k_j b_j$ such that $z - [z]_Y$ belongs to Y,

$$\{z\}_Y = z - [z]_Y \in Y \quad \text{a.e. for} \quad z \in \mathbb{R}^n, \Longrightarrow$$

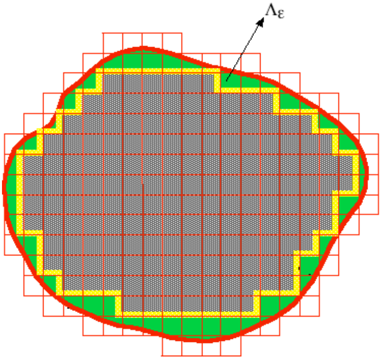
$$x = \varepsilon\left(\left[\frac{x}{\varepsilon}\right]_Y + \left\{\frac{x}{\varepsilon}\right\}_Y\right) \quad \text{a.e. for} \quad x \in \mathbb{R}^n.$$



$$\Xi_\varepsilon = \Big\{\xi \in \mathbb{Z}^N, \varepsilon(\xi + Y) \subset \Omega\Big\},$$

$$\widehat{\Omega}_\varepsilon = \text{interior}\Big\{\bigcup_{\xi\in\Xi_\varepsilon} \varepsilon\big(\xi + \overline{Y}\big)\Big\}, \quad \Lambda_\varepsilon = \Omega \setminus \widehat{\Omega}_\varepsilon.$$

$\widehat{\Omega}_\varepsilon$ is the largest union of $\varepsilon(\xi + \overline{Y})$ cells ($\xi \in \mathbb{Z}^n$) included in Ω.



Definition

For ϕ Lebesgue-measurable on Ω, $\mathcal{T}_\varepsilon$ is given by

$$\mathcal{T}_\varepsilon(\phi)(x,y) = \begin{cases} \phi\Big(\varepsilon\Big[\frac{x}{\varepsilon}\Big]_Y + \varepsilon y\Big) & \textit{a.e. for} \quad (x,y) \in \widehat{\Omega}_\varepsilon \times Y, \\ 0 & \textit{a.e. for} \quad (x,y) \in \Lambda_\varepsilon \times Y. \end{cases}$$

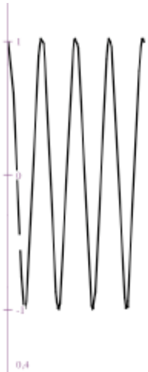
- $\mathcal{T}_\varepsilon(\phi)$ is Lebesgue-measurable,
- $\mathcal{T}_\varepsilon$ is linear and continuous from $L^p(\Omega)$ to $L^p(\Omega \times Y)$,
- $\mathcal{T}_\varepsilon(vw) = \mathcal{T}_\varepsilon(v)\,\mathcal{T}_\varepsilon(w)$,
- Let f measurable on Y, extend it by $Y-$periodicity to $\mathbb{R}^n$, and set

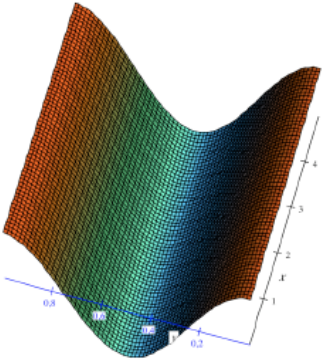
$$f_\varepsilon(x) = f\Big(\frac{x}{\varepsilon}\Big) \quad \text{a.e. for } x \in \mathbb{R}^n \Longrightarrow$$

$$\mathcal{T}_\varepsilon(f_\varepsilon|_\Omega)(x,y) = \begin{cases} f(y) & \text{a.e. for} \quad (x,y) \in \widehat{\Omega}_\varepsilon \times Y, \\ 0 & \text{a.e. for} \quad (x,y) \in \Lambda_\varepsilon \times Y. \end{cases}$$

Example 1.

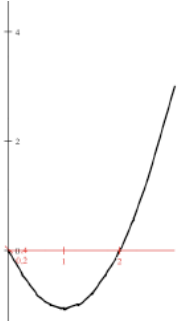
$$f(x) = \sin\left(2\pi \frac{x}{\varepsilon}\right).$$

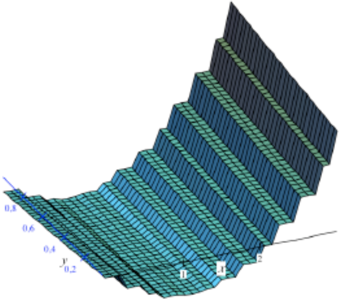




Example 2.

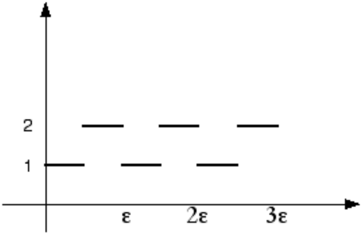
$$g(x) = x^2 - x.$$



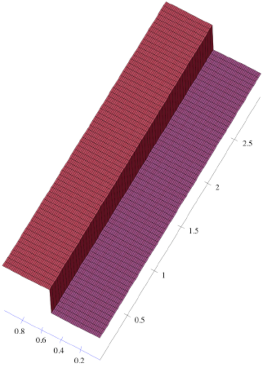


Example 3.

$$f(y) = \begin{cases} 1 & \text{for } y \in \left(0, {}^1/_2\right), \\ 2 & \text{for } y \in \left({}^1/_2, 1\right), \end{cases}$$

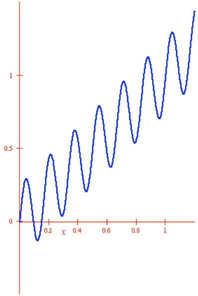


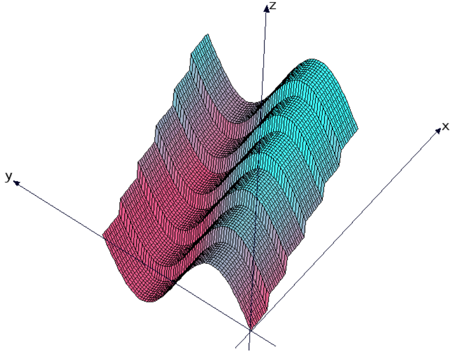
$$f_\varepsilon = f\left(\left\{\frac{x}{\varepsilon}\right\}_Y\right).$$



$\mathcal{T}_\varepsilon(f_\varepsilon)$

Example 4.





$$f_\varepsilon(x) = \frac{1}{4}\sin\left(2\pi\frac{x}{\varepsilon}\right) + x;\ \varepsilon = \frac{1}{6}$$

$$\mathcal{T}_\varepsilon(f_\varepsilon)$$

Other property. Any integral of a function on Ω, is "almost equivalent" to the integral of its unfolded on $\Omega \times Y$, the "integration defect" is due to the cells intersecting the boundary $\partial\Omega$. Indeed, for ϕ in $L^1(\Omega)$ and w in $L^p(\Omega)$

$$\frac{1}{|Y|}\int_{\Omega\times Y}\mathcal{T}_\varepsilon(\phi)(x,y)\,dx\,dy = \int_\Omega \phi(x)\,dx - \int_{\Lambda_\varepsilon}\phi(x)\,dx = \int_{\widehat{\Omega}_\varepsilon}\phi(x)\,dx,$$

$$\left|\int_\Omega \phi\,dx - \frac{1}{|Y|}\int_{\Omega\times Y}\mathcal{T}_\varepsilon(\phi)\,dxdy\right| \leq \int_{\Lambda_\varepsilon}|\phi|\,dx.$$

If $\{\phi_\varepsilon\}$ is a sequence in $L^1(\Omega)$ satisfying $\int_{\Lambda_\varepsilon}|\phi_\varepsilon|\,dx \to 0$, then

$$\int_\Omega \phi_\varepsilon\,dx - \frac{1}{|Y|}\int_{\Omega\times Y}\mathcal{T}_\varepsilon(\phi_\varepsilon)\,dxdy \to 0.$$

Notation

$$\int_\Omega \phi_\varepsilon\,dx \overset{\mathcal{T}_\varepsilon}{\simeq} \frac{1}{|Y|}\int_{\Omega\times Y}\mathcal{T}_\varepsilon(\phi_\varepsilon)\,dxdy.$$

2. First convergence results

Let p belong to $[1, +\infty[$.

$$\mathcal{T}_\varepsilon(w) \to w \quad \text{strongly in } L^p(\Omega \times Y), \quad \forall w \in L^p(\Omega).$$

Let $\{w_\varepsilon\}$ be a sequence in $L^p(\Omega)$ such that $w_\varepsilon \to w$ strongly in $L^p(\Omega)$. Then

$$\mathcal{T}_\varepsilon(w_\varepsilon) \to w \quad \text{strongly in } L^p(\Omega \times Y).$$

For every relatively weakly compact sequence $\{w_\varepsilon\} \subset L^p(\Omega)$, $\mathcal{T}_\varepsilon(w_\varepsilon)$ is relatively weakly compact in $L^p(\Omega \times Y)$. Furthermore, if $\mathcal{T}_\varepsilon(w_\varepsilon) \rightharpoonup \widehat{w}$ weakly in $L^p(\Omega \times Y)$, then

$$w_\varepsilon \rightharpoonup \mathcal{M}_Y(\widehat{w}) = \frac{1}{|Y|} \int_Y \widehat{w} \, dy \quad \text{weakly in } L^p(\Omega).$$

Two-scale convergence. Let $p \in]1, +\infty[$. A bounded sequence $\{w_\varepsilon\}$ in $L^p(\Omega)$, two-scale converges to some w belonging to $L^p(\Omega \times Y)$, whenever, for every smooth function ϕ on $\Omega \times Y$, the following convergence holds :

$$\int_\Omega w_\varepsilon(x)\phi\Big(x, \frac{x}{\varepsilon}\Big)\, dx \to \frac{1}{|Y|} \int\int_{\Omega\times Y} w(x,y)\,\phi(x,y)\, dxdy.$$

The following result holds true :

If $\{w_\varepsilon\}$ is a bounded sequence in $L^p(\Omega)$ with $p \in]1, +\infty[$, then the following assertions are equivalent :

- $\{\mathcal{T}_\varepsilon(w_\varepsilon)\}$ converges weakly to w in $L^p(\Omega \times Y)$.
- $\{w_\varepsilon\}$ two-scale converges to w.

Remark. To check the two-scale convergence one has to use special test functions. To check a weak convergence in the space $L^p(\Omega \times Y)$ due to density, one can use of functions smooth functions from $\mathcal{D}(\Omega \times Y)$.

3. Macro-micro decomposition : the scale-splitting operators $\mathcal{Q}_\varepsilon$ and $\mathcal{R}_\varepsilon$

Every ϕ in $W_0^{1,p}(\Omega)$ in the form

$$\phi = \mathcal{Q}_\varepsilon(\phi) + \mathcal{R}_\varepsilon(\phi),$$

where $\mathcal{Q}_\varepsilon(\phi)$ is an approximation of ϕ having the same behavior as ϕ, while $\mathcal{R}_\varepsilon(\phi)$ is a remainder of order ε.
Suppose that $\partial\Omega$ is smooth enough so that there exists a continuous extension operator $\mathcal{P} : W^{1,p}(\Omega) \mapsto W^{1,p}(\mathbb{R}^n)$ satisfying

$$\|\mathcal{P}(\phi)\|_{W^{1,p}(\mathbb{R}^n)} \leq C \|\phi\|_{W^{1,p}(\Omega)}, \quad \forall \phi \in W^{1,p}(\Omega),$$

where C is a constant depending on p and $\partial\Omega$ only.
The construction of $\mathcal{Q}_\varepsilon$ is based on the Q_1-interpolate of a discrete approximation, as in the finite element method.

The operator $\widetilde{\mathcal{Q}_\varepsilon} : L^p(\mathbb{R}^n) \mapsto W^{1,\infty}(\mathbb{R}^n)$, for $p \in [1, +\infty]$, is defined as follows

$$\widetilde{\mathcal{Q}_\varepsilon}(\phi)(\varepsilon\xi) = \mathcal{M}_\varepsilon(\phi)(\varepsilon\xi) \quad \text{for } \xi \in \varepsilon\mathbb{Z}^n,$$

where the local average operator $\mathcal{M}_\varepsilon$ is defined by

$$\mathcal{M}_\varepsilon(\phi)(x) = \begin{cases} \dfrac{1}{\varepsilon^n|Y|} \displaystyle\int_{\varepsilon\left[\frac{x}{\varepsilon}\right] + \varepsilon Y} \phi(\zeta)\, d\zeta, & \text{if } x \in \widehat{\Omega}_\varepsilon, \\ 0 \qquad \text{if } x \in \Lambda_\varepsilon. \end{cases}$$

Observe that for every $\phi \in L^p(\Omega)$,

$$\mathcal{M}_\varepsilon(\phi) \to \phi \quad \text{strongly in } L^p(\Omega).$$

For any $x \in \mathbb{R}^n$,

$\widetilde{\mathcal{Q}_\varepsilon}(\phi)(x)$ is the Q_1 interpolate of the values of $\widetilde{\mathcal{Q}_\varepsilon}(\phi)$ at the vertices of the cell $\varepsilon\left[\dfrac{x}{\varepsilon}\right]_Y + \varepsilon Y$.

Then the operator $\mathcal{Q}_\varepsilon : W^{1,p}(\Omega) \mapsto W^{1,\infty}(\Omega)$ is defined by

$$\mathcal{Q}_\varepsilon(\phi) = \widetilde{\mathcal{Q}_\varepsilon}(\mathcal{P}(\phi))|_\Omega.$$

Well known (from finite elements method)

- $\widetilde{\mathcal{Q}_\varepsilon}(\phi) \to \phi$ strongly in $L^p(\mathbb{R}^n)$,
- $\varepsilon\nabla\widetilde{\mathcal{Q}_\varepsilon}(\phi) \to 0$ strongly in $(L^p(\mathbb{R}^n))^n$,
- $\|\mathcal{Q}_\varepsilon(\phi)\|_{W^{1,p}(\Omega)} \leq C\,\|\mathcal{P}\|\|\phi\|_{W^{1,p}(\Omega)}$.

Clear : $\mathcal{Q}_\varepsilon$ is designed NOT to capture any oscillation of order ε.

The remainder $\mathcal{R}_\varepsilon(\phi)$ is given by

$$\mathcal{R}_\varepsilon(\phi) = \phi - \mathcal{Q}_\varepsilon(\phi) \quad \text{for any } \phi \in W^{1,p}(\Omega),$$

and has the following properties :

$$\begin{aligned} \|\mathcal{R}_\varepsilon(\phi)\|_{L^p(\Omega)} &\leq \varepsilon\, C\, \|\mathcal{P}\| \|\phi\|_{W^{1,p}(\Omega)}, \\ \|\nabla \mathcal{R}_\varepsilon(\phi)\|_{L^p(\Omega)} &\leq \; C\, \|\mathcal{P}\| \|\nabla \phi\|_{L^p(\Omega)} \end{aligned}$$

Let $\{w_\varepsilon\}$ be a sequence converging weakly in $W^{1,p}(\Omega)$ to w. Then,

$$\mathcal{R}_\varepsilon(w_\varepsilon) \to 0 \quad \textit{strongly in } L^p(\Omega),$$

$$\mathcal{Q}_\varepsilon(w_\varepsilon) \rightharpoonup w \quad \textit{weakly in } W^{1,p}(\Omega),$$

$$\mathcal{T}_\varepsilon(\nabla \mathcal{Q}_\varepsilon(w_\varepsilon)) \rightharpoonup \nabla w \quad \textit{weakly in } L^p(\Omega \times Y).$$

4. Main convergence result

Theorem

Let $\{w_\varepsilon\} \subset W^{1,p}(\Omega)$ *satisfying*

$$w_\varepsilon \rightharpoonup w \quad \text{weakly in } W^{1,p}(\Omega).$$

Then, up to a subsequence there exists some $\widehat{w}$ *in the space* $L^p(\Omega; W^{1,p}_{per}(Y))$ *such that*

$$\bullet \frac{1}{\varepsilon}\mathcal{T}_\varepsilon\big(\mathcal{R}_\varepsilon(w_\varepsilon)\big) \rightharpoonup \widehat{w} \quad \text{weakly in } L^p(\Omega; W^{1,p}(Y)),$$

$$\bullet \mathcal{T}_\varepsilon\big(\nabla\mathcal{R}_\varepsilon(w_\varepsilon)\big) \rightharpoonup \nabla_y\widehat{w} \quad \text{weakly in } L^p(\Omega \times Y),$$

$$\bullet \mathcal{T}_\varepsilon(\nabla w_\varepsilon) \rightharpoonup \nabla w + \nabla_y\widehat{w} \quad \text{weakly in } L^p(\Omega \times Y).$$

5. Periodic unfolding and the standard homogenization

Let $A^\varepsilon = (a^\varepsilon_{ij})_{1\le i,j\le n}$ be a sequence of matrices satisfying a.e. on Ω

$$(A^\varepsilon\lambda,\lambda) \ge \alpha|\lambda|^2, \qquad |A^\varepsilon(x)\lambda| \le \beta|\lambda|, \quad \forall\lambda\in\mathbb{R}^n.$$

For f given in $H^{-1}(\Omega)$, consider the Dirichlet problem

$$\begin{cases} -\text{div}\,(A^\varepsilon\nabla u_\varepsilon) = f \quad \text{in } \Omega, \\ u^\varepsilon = 0 \quad \text{on } \partial\Omega. \end{cases}$$

By the Lax-Milgram theorem, there exists a unique $u^\varepsilon \in H^1_0(\Omega)$

$$\int_\Omega A^\varepsilon\nabla u_\varepsilon\,\nabla v\,dx = \langle f, v\rangle_{H^{-1}(\Omega),H^1_0(\Omega)}, \quad \forall v\in H^1_0(\Omega).$$

Moreover, one has the apriori estimate

$$\|u_\varepsilon\|_{H^1_0(\Omega)} \le \frac{1}{\alpha}\|f\|_{H^{-1}(\Omega)} \Longrightarrow u_\varepsilon \rightharpoonup u_0 \quad \text{weakly in } H^1_0(\Omega).$$

Classical periodic homogenization result. Let $a_{ij} = a_{ij}(y)$ for $1 \leq i,j \leq n$, be $Y-$periodic, and let $A^\varepsilon(x) = (a_{ij}(x/\varepsilon))_{1\leq i,j\leq n}$ a.e. on Ω. Then u_0 is the unique solution of

$$\begin{cases} -\text{div}\,(A^0 \nabla u_0) = \displaystyle\sum_{i,j=1}^{n} a^0_{ij} \frac{\partial^2 u_0}{\partial x_i \partial x_j} = f \quad \text{in } \Omega, \\ u_0 = 0 \quad \text{on } \partial\Omega, \end{cases}$$

where the constant matrix $A^0 = (a^0_{ij})_{1\leq i,j\leq n}$ is elliptic and given by

$$a^0_{ij} = \mathcal{M}_Y(a_{ij}) - \mathcal{M}_Y\Big(\sum_{k=1}^{n} a_{ik} \frac{\partial \widehat{\chi}^j}{\partial y_k}\Big).$$

The corrector function χ^j for $\forall j \in \{1,\ldots,n\}$, is solution of

$$\begin{cases} -\displaystyle\sum_{i,k=1}^{n} \frac{\partial}{\partial y_i}\left(a_{ik} \frac{\partial(\widehat{\chi}^j - y_j)}{\partial y_k}\right) = 0 \quad \text{in } Y, \\ \widehat{\chi}^j \quad Y\text{-periodic and } \mathcal{M}_Y(\widehat{\chi}^j) = 0. \end{cases}$$

Periodic homogenization via unfolding. Suppose that there exists a matrix B such that

$$B^\varepsilon = \mathcal{T}_\varepsilon\big(A^\varepsilon\big) \to B \quad \text{strongly in } [L^1(\Omega \times Y)]^{n\times n}.$$

Then there exists $u_0 \in H^1_0(\Omega)$ and $\widehat{u} \in L^2(\Omega; H^1_{per}(Y))$ such that

$$\begin{cases} u_\varepsilon \rightharpoonup u_0 \quad \text{weakly in } H^1_0(\Omega), \\ \mathcal{T}_\varepsilon(\nabla u_\varepsilon) \rightharpoonup \nabla u_0 + \nabla_y \widehat{u} \quad \text{weakly in } L^2(\Omega \times Y), \end{cases}$$

and the pair $(u_0, \widehat{u})$ is the unique solution of the problem

$$\begin{cases} \forall \Psi \in H^1_0(\Omega),\ \forall \Phi \in L^2(\Omega;\ H^1_{per}(Y)), \\ \dfrac{1}{|Y|}\displaystyle\int_{\Omega\times Y} B(x,y)\big[\nabla u_0(x) + \nabla_y \widehat{u}(x,y)\big]\big[\nabla\Psi(x) + \nabla_y\Phi(x,y)\big]\,dxdy \\ \qquad\qquad = \ \langle f, \Psi\rangle_{H^{-1}(\Omega),H^1_0(\Omega)} \end{cases}$$

If $A^\varepsilon(x) = (A(x/\varepsilon)$, then $B(x,y) = A(y)$.

The proof by the periodic unfolding method is elementary !

Integration formula $\Longrightarrow$ With $\Psi \in H^1_0(\Omega)$,

$$\langle f, \Psi \rangle_{H^{-1}(\Omega), H^1_0(\Omega)} = \int_\Omega A^\varepsilon \nabla u_\varepsilon \, \nabla \Psi \; dx$$

$$\overset{\mathcal{T}_\varepsilon}{\simeq} \frac{1}{|Y|} \int_{\Omega\times Y} \mathcal{T}_\varepsilon(A^\varepsilon)\, \mathcal{T}_\varepsilon\big(\nabla u_\varepsilon\big) \mathcal{T}_\varepsilon\big(\nabla \Psi\big)\, dx\, dy$$

$$\frac{1}{|Y|} \int_{\Omega\times Y} B_\varepsilon\, \mathcal{T}_\varepsilon\big(\nabla u_\varepsilon\big) \mathcal{T}_\varepsilon\big(\nabla \Psi\big)\, dx\, dy$$

and the last integral gives at the limit

$$\frac{1}{|Y|} \int_{\Omega\times Y} B(x,y) \big[\nabla u_0(x) + \nabla_y \widehat{u}(x,y)\big] \nabla \Psi(x) \, dxdy = \langle f, \Psi \rangle_{H^{-1}(\Omega), H^1_0(\Omega)}.$$

Take now as test function

$$v^\varepsilon(x) = \varepsilon\Psi(x)\psi\Big(\frac{x}{\varepsilon}\Big),\ \Psi \in \mathcal{D}(\Omega),\ \psi \in H^1_{per}(Y).$$

$$\frac{1}{|Y|}\int_{\Omega\times Y} B^\varepsilon\ \mathcal{T}_\varepsilon\big(\nabla u_\varepsilon\big)\ \varepsilon\psi(y)\mathcal{T}_\varepsilon\big(\nabla\Psi\big)\ dxdy$$
$$+\frac{1}{|Y|}\int_{\Omega\times Y} B^\varepsilon\ \mathcal{T}_\varepsilon\big(\nabla u_\varepsilon\big)\nabla_y\psi(y)\mathcal{T}_\varepsilon\big(\Psi\big)\ dxdy \overset{\mathcal{T}_\varepsilon}{\simeq}\ \langle f, v_\varepsilon\rangle_{H^{-1}(\Omega),H^1_0(\Omega)}.$$

Since $v^\varepsilon \rightharpoonup 0$ in $H^1_0(\Omega)$, passing to the limit one has

$$\frac{1}{|Y|}\int_{\Omega\times Y} B(x,y)\big[\nabla u_0(x) + \nabla_y\widehat{u}(x,y)\big]\Psi(x)\nabla_y\psi(y)\ dxdy = 0.$$

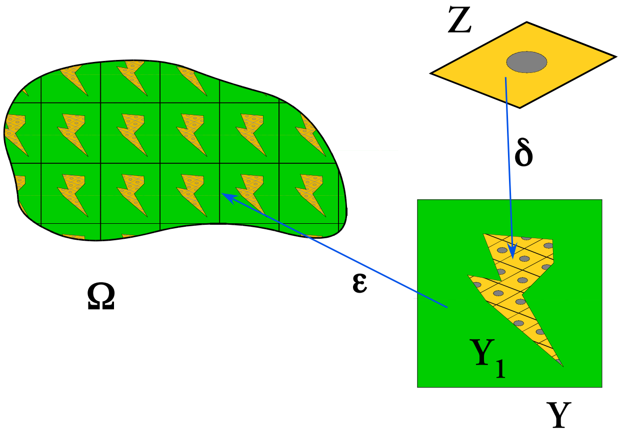
By the density of the tensor product $\mathcal{D}(\Omega)\otimes H^1_{per}(Y)$ in $L^2(\Omega; H^1_{per}(Y))$, this holds for all Φ in $L^2(\Omega; H^1_{per}(Y))$. □

II. Periodic unfolding and multiscales

The periodic unfolding method turns out to be particularly well-adapted to multi-scales problems.
Consider two periodicity cells Y and Z. Suppose that Y is"partitioned" in two non-empty disjoint open subsets Y_1 and Y_2, i.e. such that $Y_1 \cap Y_2 = \emptyset$ and $\overline{Y} = \overline{Y}_1 \cup \overline{Y}_2$.
Let $A^{\varepsilon\delta}$ be a matrix field defined by

$$A^{\varepsilon\delta}(x) = \begin{cases} A_1\Big(\Big\{\dfrac{x}{\varepsilon}\Big\}_Y\Big) & \text{for} \quad \Big\{\dfrac{x}{\varepsilon}\Big\}_Y \in Y_1 \\ A_2\Bigg(\Bigg\{\dfrac{\big\{\frac{x}{\varepsilon}\big\}_Y}{\delta}\Bigg\}_Z\Bigg) & \text{for} \quad \Big\{\dfrac{x}{\varepsilon}\Big\}_Y \in Y_2, \end{cases}$$

where A_1 and A_2 have the same properties as before. Here there are two small scales, ε and $\varepsilon\delta$, associated respectively to the cells Y and Z.


Z
δ
ε
Ω
Y_1
Y

Consider the problem

$$\int_\Omega A^{\varepsilon\delta}\nabla u_{\varepsilon\delta}\nabla w \; dx = \int_\Omega f \; w \; dx, \qquad \forall w \in H_0^1(\Omega),$$

with f in $L^2(\Omega)$. The Lax-Milgram theorem gives the existence and uniqueness of $u_{\varepsilon\delta}$ in $H_0^1(\Omega)$ satisfying the estimate

$$\|u_{\varepsilon\delta}\|_{H_0^1(\Omega)} \leq \frac{1}{\alpha}\|f\|_{L^2(\Omega)}.$$

So, as above there is some $u_0 \in H_0^1(\Omega)$ and $\widehat{u} \in L^2(\Omega; H^1_{per}(Y))$ such that,

$$u_{\varepsilon\delta} \rightharpoonup u_0 \quad \text{weakly in } H_0^1(\Omega),$$

and

$$\mathcal{T}_\varepsilon\big(\nabla u_{\varepsilon\delta}\big) \rightharpoonup \nabla u_0 + \nabla_y \widehat{u} \quad \text{weakly in } L^2(\Omega \times Y).$$

These convergences do not see the oscillations at the scale $\varepsilon\delta$.

In order to capture them, one considers the restrictions to the set $\Omega \times Y_2$

$$v_{\varepsilon\delta}(x,y) \doteq \frac{1}{\varepsilon}\mathcal{T}_\varepsilon\big(\mathcal{R}_\varepsilon(u_{\varepsilon\delta})\big)|_{Y_2}.$$

Obviously,

$$v_{\varepsilon\delta} \rightharpoonup \widehat{u}|_{Y_2} \quad \text{weakly in } \; L^2(\Omega; H^1(Y_2)).$$

Apply to $v_{\varepsilon\delta}$ a similar unfolding operation, denoted $\mathcal{T}_\delta^y$, for the variable y, thus adding a new variable $z \in Z$.

$$\mathcal{T}_\delta^y(v_{\varepsilon\delta})(x,y,z) = v_{\varepsilon\delta}\Big(x, \delta\Big[\frac{y}{\delta}\Big]_Z + \delta z\Big) \quad \text{for } x \in \Omega, \;\; y \in Y_2 \text{ and } z \in Z.$$

All the estimates and weak convergence properties for the original unfolding $\mathcal{T}_\varepsilon$ still hold for $\mathcal{T}_\delta^y$ with x being a mere parameter. $\Longrightarrow$
There is $\widetilde{u} \in L^2(\Omega \times Y_2, H^1_{per}(Z)/\mathbb{R})$ such that

$$\mathcal{T}_\delta^y\big(\nabla_y v_{\varepsilon\delta}\big) \rightharpoonup \nabla_y \widehat{u}|_{\Omega_2} + \nabla_z \widetilde{u} \quad \text{weakly in } L^2(\Omega \times Y_2 \times Z),$$

$$\mathcal{T}_\delta^y\Big(\mathcal{T}_\varepsilon\big(\nabla u_{\varepsilon\delta}\big)\Big) \rightharpoonup \nabla u_0 + \nabla_y \widehat{u} + \nabla_z \widetilde{u} \quad \text{weakly in } L^2(\Omega \times Y_2 \times Z).$$

Theorem. The functions

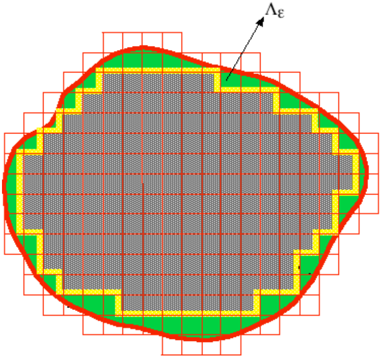
$$u_0 \in H_0^1(\Omega), \quad \widehat{u} \in L^2(\Omega, H^1_{per}(Y)/\mathbb{R}), \quad \widetilde{u} \in L^2(\Omega\times Y_2, H^1_{per}(Z)/\mathbb{R})$$

are the unique solutions of the variational problem

$$\begin{cases} \dfrac{1}{|Y||Z|} \displaystyle\int_\Omega \int_{Y_2} \int_Z A_2(z)\Big[\nabla u_0 + \nabla_y \widehat{u} + \nabla_z \widetilde{u}\Big]\Big[\nabla \Psi + \nabla_y \Phi + \nabla_z \Theta\Big] dxdydz \\ \qquad + \dfrac{1}{|Y|} \displaystyle\int_\Omega \int_{Y_1} A_1(y)\Big[\nabla u_0 + \nabla_y \widehat{u}\Big]\Big[\nabla \Psi + \nabla_y \Phi\Big]\, dxdy = \int_\Omega f\, \Psi\, dx, \\ \forall \Psi \in H_0^1(\Omega),\ \forall \Phi \in L^2(\Omega;\, H^1_{per}(Y)/\mathbb{R}),\ \forall \Theta \in L^2(\Omega \times \Omega_2, H^1_{per}(Z)/\mathbb{R}). \end{cases}$$

Perforated domains

Same notations

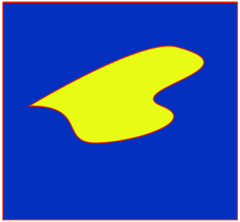


$$\Xi_\varepsilon = \left\{ \xi \in \mathbb{Z}^N, \varepsilon(\xi + Y) \subset \Omega \right\},$$

$$\widehat{\Omega}_\varepsilon = \text{interior}\left\{ \bigcup_{\xi \in \Xi_\varepsilon} \varepsilon\big(\xi + \overline{Y}\big) \right\}, \quad \Lambda_\varepsilon = \Omega \setminus \widehat{\Omega}_\varepsilon.$$

The notion of periodig unfolding is well adapted for perforated domains, almost all the results of concerning fixed domains still hold.

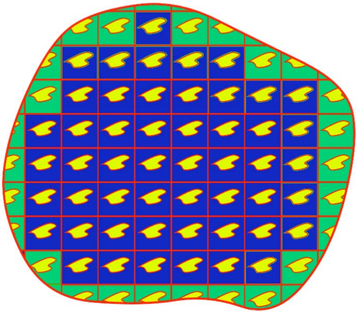
Let S be a strict closed subset of $\overline{Y}$ and set $Y^* = Y \setminus S$ (the part occupied by the material).



Here Y happens to be a parallelotop, but the definition holds for a general Y.

The perforated domain Ω^*_ε is obtained by removing from Ω the set of holes S_ε

$$\Omega^*_\varepsilon = \Omega \setminus S_\varepsilon, \qquad \text{where } \; S_\varepsilon = \bigcup_{\xi \in \mathbf{G}} \varepsilon(\xi + S).$$



The sets Ω^*_ε (blue and green), $\widehat{\Omega}^*_\varepsilon$ (blue) and Λ^*_ε (green).

Definition For any function ϕ Lebesgue-measurable on Ω_ε^*, the unfolding operator $\mathcal{T}_\varepsilon^*$ is defined by

$$\mathcal{T}_\varepsilon^*(\phi)(x,y) = \begin{cases} \phi\Big(\varepsilon\Big[\frac{x}{\varepsilon}\Big]_Y + \varepsilon y\Big) & \text{a.e. for} \quad (x,y) \in \widehat{\Omega}_\varepsilon \times Y^*, \\ 0 & \text{a.e. for} \quad (x,y) \in \Lambda_\varepsilon \times Y^*. \end{cases}$$

The relationship between $\mathcal{T}_\varepsilon$ and $\mathcal{T}_\varepsilon^*$ is given for any w defined on Ω_ε^*, by

$$\mathcal{T}_\varepsilon^*(w) = \mathcal{T}_\varepsilon(\widetilde{w})_{|_{\Omega \times Y^*}}.$$

$\Longrightarrow$ The operator $\mathcal{T}_\varepsilon^*$ enjoys properties which follow directly from those of $\mathcal{T}_\varepsilon$.

The main convergence result requires that the perforated cell Y^* satisfy the following geometrical hypothesis :

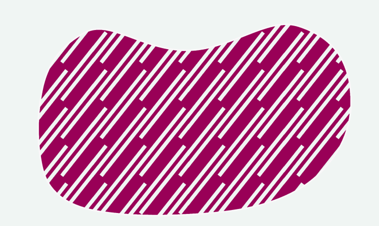
The open set Y^* satisfies the Poincaré-Wirtinger inequality for the exponent p ($p \in [1, +\infty]$) and for all vector b_i, $i \in \{1, \dots, n\}$ of the periodicity basis, the open set interior$(\overline{Y^* \cup (b_i + Y^*)})$ is connected.

The bounded open set $\mathcal{O}$ satisfies the Poincaré-Wirtinger inequality for the exponent $p \in [1, +\infty]$ if there exists a constant C_p such that

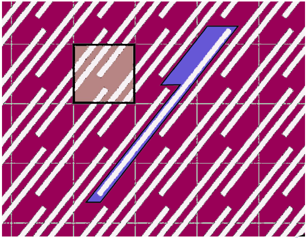
$$\forall u \in W^{1,p}(\mathcal{O}), \quad \|u - \mathcal{M}_{\mathcal{O}}(u)\|_{L^p(\mathcal{O})} \leq C_p \|\nabla u\|_{L^p(\mathcal{O})}.$$

If the reference hole S is compact in a parallelotop Y, and if Y^* satisfies the Poincaré-Wirtinger inequality for $p \in [1, +\infty]$, then the geometrical hypothesis holds true.

In general, the cell Y is not a parallelotop, and this requires the use of another reference cell with a more general shape.



No choice of parallelotop gives a connected Y^*, while there are many possible Y's that give a connected Y^*.



Main theorem. Let w_ε in $W^{1,p}(\Omega_\varepsilon^*)$ satisfying

$$\|\nabla w_\varepsilon\|_{(L^p(\Omega_\varepsilon^*))^n} \leq C.$$

Then (up tp a subsequence), there exist w in $W^{1,p}(\Omega)$ and $\widehat{w}$ in $L^p(\Omega; W^{1,p}_{per}(Y^*))$,

$$\begin{aligned}
&\mathcal{T}_\varepsilon^*(w_\varepsilon) \to w \quad \text{strongly in } L^p_{loc}(\Omega; W^{1,p}(Y^*)),\\
&\mathcal{T}_\varepsilon^*(w_\varepsilon) \rightharpoonup w \quad \text{weakly in } L^p(\Omega; W^{1,p}(Y^*)),\\
&\mathcal{T}_\varepsilon^*(\nabla w_\varepsilon) \rightharpoonup \nabla w + \nabla_y \widehat{w} \quad \text{weakly in } L^p(\Omega \times Y^*).
\end{aligned}$$

Classical homogenization in periodically perforated domains

One can treat a full range of problems without asking nor that the holes do not intersect $\partial\Omega$, neither that ∂S be sufficiently smooth (necessary for having extension operators to the whole of Ω). It is again elementary to homogenize the Dirichlet-Neumann problem

$$\begin{cases} -\text{div}\,(A^{\varepsilon}\nabla u_{\varepsilon}) = f \quad \text{in } \Omega^*_{\varepsilon}, \\ u_{\varepsilon} = 0 \quad \text{on } \partial\Omega \cap \partial\Omega^*_{\varepsilon}, \\ A^{\varepsilon} u_{\varepsilon} \cdot n_{\varepsilon} = 0 \quad \text{on } \partial S_{\varepsilon} \cap \partial\Omega^*_{\varepsilon}. \end{cases}$$

and the Neumann problem

$$\begin{cases} -\text{div}\,(A^{\varepsilon}\nabla u_{\varepsilon}) + u_{\varepsilon} = f \quad \text{in } \Omega^*_{\varepsilon}, \\ A^{\varepsilon} u_{\varepsilon} \cdot n_{\varepsilon} = 0 \quad \text{on } \partial\Omega^*_{\varepsilon}. \end{cases}$$

For both problems one has

$$\widetilde{u}_{\varepsilon} \rightharpoonup \frac{|Y^*|}{|Y|}\, u_0 \quad \text{weakly in } L^2(\Omega),$$

with u_0 solution of the corresponding homogenized problem.

Robin boundary conditions on the boundary of the holes

In the same framework, consider the following non-homogeneous Neumann problem, :

$$\begin{cases} -\text{div}\,(A^\varepsilon \nabla u_\varepsilon) = f \quad \text{in } \Omega_\varepsilon^*, \\ u_\varepsilon = 0 \quad \text{on } \partial\Omega_\varepsilon^* \cap \partial\Omega, \\ A^\varepsilon u_\varepsilon \cdot n_\varepsilon = g_\varepsilon \quad \text{on } \partial S_\varepsilon \cap \Omega. \end{cases} \tag{6.1}$$

The domain Ω is bounded with Lipschitz boundary $\partial\Omega$. The function g_ε is given in $L^2(\partial S_\varepsilon \cap \Omega)$. The variational formulation is

$$\begin{cases} \text{Find } u_\varepsilon \in H_0^1(\Omega_\varepsilon^*; \partial\Omega \cap \partial\Omega_\varepsilon^*) \text{ such that} \\ \displaystyle\int_{\Omega_\varepsilon^*} A^\varepsilon \nabla u_\varepsilon \nabla v \, dx = \int_{\Omega_\varepsilon^*} f\, v\, dx + \int_{\partial S_\varepsilon \cap \Omega} g_\varepsilon v \, d\sigma(x), \\ \forall v \in H_0^1(\Omega_\varepsilon^*; \partial\Omega \cap \partial\Omega_\varepsilon^*). \end{cases}$$

Difficulty : the treatment of the surface integral.

The boundary unfolding operator

For p be in $(1,+\infty)$, suppose that ∂S has a finite number of connected components and that ∂S is Lipschitz. The boundary of the set of holes in Ω is $\partial S_\varepsilon \cap \Omega$. Then a well-defined trace operator exists from $W^{1,p}(\widehat{\Omega}^*_\varepsilon)$ to $W^{1-1/p,p}(\widehat{\partial S}_\varepsilon)$.

Definition For any function φ Lebesgue-measurable on $\partial\widehat{\Omega}^*_\varepsilon \cap \partial S^\varepsilon$, the boundary unfolding operator $\mathcal{T}^b_\varepsilon$ is defined by

$$\mathcal{T}^b_\varepsilon(\phi)(x,y) = \begin{cases} \phi\Big(\varepsilon\Big[\dfrac{x}{\varepsilon}\Big]_Y + \varepsilon y\Big) & \text{a.e. for } \quad (x,y) \in \widehat{\Omega}_\varepsilon \times \partial S, \\ 0 & \text{a.e. for } \quad (x,y) \in \Lambda_\varepsilon \times \partial S. \end{cases}$$

If $\varphi \in W^{1,p}(\Omega^*_\varepsilon)$, $\mathcal{T}^b_\varepsilon(\varphi)$ is just the trace on ∂S of $\mathcal{T}^*_\varepsilon(\varphi)$.

The operator $\mathcal{T}_\varepsilon^b$ has similar properties as the previous unfolding operators. In particular, the integration formula reads

$$\frac{1}{\varepsilon|Y|}\int_{\Omega\times\partial S}\mathcal{T}_\varepsilon^b(\varphi)(x,y)\,dx\,d\sigma(y)=\int_{\widehat{\partial S_\varepsilon}}\varphi(x)\,d\sigma(x).$$

One can now give several convergence results allowing to give the limit of the integral

$$\int_{\partial S_\varepsilon\cap\Omega} g_\varepsilon v\,d\sigma(x),\ \text{with } v\in H_0^1(\Omega).$$

To do so, one has to make some hypothese on the behaviour of g_ε.

Example.
If there exist g in $L^2(\Omega \times \partial S)$ and G in $L^2(\Omega)$ satisfying

$$\mathcal{T}_\varepsilon^b(g_\varepsilon) \rightharpoonup g \quad \text{weakly in } L^2(\Omega \times \partial S),$$

$$\frac{1}{\varepsilon}\mathcal{M}_{\partial S}(\mathcal{T}_\varepsilon^b(g_\varepsilon)) \rightharpoonup G \quad \text{weakly in } L^2(\Omega),$$

then the limit of $\int_{\partial S_\varepsilon \cap \Omega} g_\varepsilon v \, d\sigma(x)$ is

$$\frac{|\partial S|}{|Y|}\int_\Omega \mathcal{M}_{\partial S}(y_{\mathcal{M}} g)(x) \cdot \nabla v(x)\, dx + \frac{|\partial S|}{|Y|}\int_\Omega G(x) v(x)\, dx,$$

where $y_{\mathcal{M}} = y - \mathcal{M}_{Y^*}(y)$.

Multi-scale domains

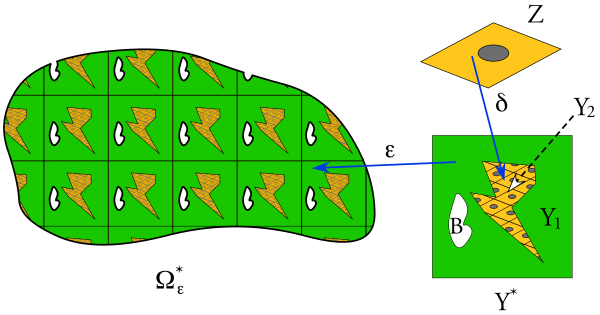
The unfolding methods for fixed domains and for perforated domains, can be combined to consider mixed situations.
Let Y^* be a subset of Y, and Y_2 be given an open subset of Y^* with Lipschitz boundary. Denote $Y^* \setminus Y_2$ by Y_1. Let Z be another periodicity cell, and ε, ∂ be two small parameters.
For x in $(\mathbb{R}^n)^*_\varepsilon$, set $A^{\varepsilon\delta}$ be a matrix field defined by

$$A^{\varepsilon\delta}(x) = \begin{cases} A_1\Big(\Big\{\dfrac{x}{\varepsilon}\Big\}_Y\Big) & \text{for} \quad \Big\{\dfrac{x}{\varepsilon}\Big\}_Y \in Y_1, \\ A_2\Big(\Big\{\dfrac{\{\frac{x}{\varepsilon}\}_Y}{\delta}\Big\}_Z\Big) & \text{for} \quad \Big\{\dfrac{x}{\varepsilon}\Big\}_Y \in Y_2. \end{cases}$$

As before, the perforated domain Ω_ε^* is defined by

$$\Omega_\varepsilon^* = \Omega \setminus \overline{S}_\varepsilon.$$

So, Ω_ε^* has $\varepsilon-$periodic holes (the set S_ε) and a $\varepsilon-$periodic set of a composite material corresponding to the set $Y_{2,\varepsilon}$.



The homogenized problem corresponding to

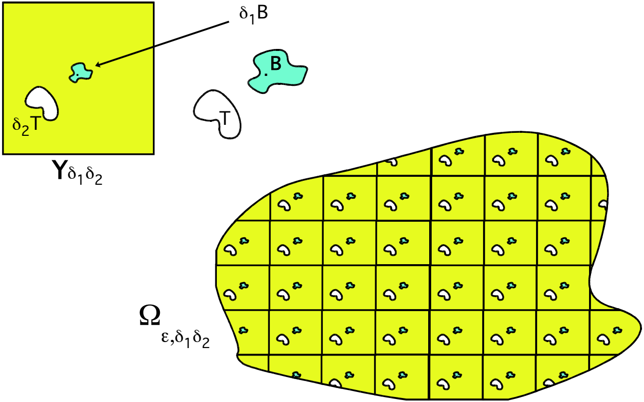
$$\int_{\Omega_\varepsilon^*} A^{\varepsilon\delta}\nabla u_{\varepsilon\delta}\nabla v\,dx = \int_{\Omega_\varepsilon^*} f\,v\,dx, \qquad \forall v \in H_0^1(\Omega_\varepsilon^*; \partial\Omega \cap \partial\Omega_\varepsilon^*),$$

is of the form

$$\frac{1}{|Y||Z|}\int_\Omega\int_{Y_2}\int_Z A_2(z)\Big[\nabla u_0 + \nabla_y \widehat{u} + \nabla_z \widehat{u}_1\Big]\Big[\nabla\Psi + \nabla_y\Phi + \nabla_z\Theta\Big]\,dxdydz$$
$$+\frac{1}{|Y|}\int_\Omega\int_{Y_1} A_1(y)\Big[\nabla u_0 + \nabla_y\widehat{u}\Big]\Big[\nabla\Psi + \nabla_y\Phi\Big]\,dxdy = \frac{|Y^*|}{|Y|}\int_\Omega f\,\Psi\,dx,$$

for $\Psi \in H_0^1(\Omega)$, $\Phi \in L^2(\Omega;\ H^1_{per}(Y^*)/\mathbb{R})$ and
$\Theta \in L^2(\Omega \times Y_2; H^1_{per}(Z)/\mathbb{R})$.

Domains with "small" holes



To treat such a situation one defines an appropriate unfolding operator depending on several small parameters.

Reticulated structures