An introduction to Isogeometric Analysis

A. Buffa[1], G. Sangalli[1,2], R. Vázquez[1]

[1] Istituto di Matematica Applicata e Tecnologie Informatiche - Pavia

Consiglio Nazionale delle Ricerche

[2] Dipartimento di Matematica, Università di Pavia

Supported by the ERC Starting Grant:
GeoPDEs n. 205004

1. Approximation of vector fields and differential forms
    - Construction of the discrete spaces
    - The commuting De Rham diagram
    - Maxwell eigenproblem: B-splines discretization
    - Maxwell eigenproblem: NURBS discretization

Part I

Approximation of vector fields and differential forms

Setting of the problem

Let Ω be a Lipschitz domain described by a NURBS mapping

$$\mathbf{F} : \widehat{\Omega} \longrightarrow \Omega, \quad \widehat{\Omega} \text{ being the unit cube.}$$

We define the functional spaces

$$H(\mathsf{d}, \Omega) = \left\{ u \in L^2(\Omega) \ : \ \mathsf{d} u \in L^2(\Omega) \right\}, \quad \|u\|_{\mathsf{d},\Omega} = \|u\|_0 + \|\mathsf{d}u\|_0$$

$$H^1(\Omega)/\mathbb{R} \xrightarrow{\ \mathbf{grad}\ } H(\mathbf{curl}\,, \Omega) \xrightarrow{\ \mathbf{curl}\ } H(\mathrm{div}, \Omega) \xrightarrow{\ \mathrm{div}\ } L^2(\Omega).$$

Setting of the problem

Let Ω be a Lipschitz domain described by a NURBS mapping

$$\mathbf{F} : \widehat{\Omega} \longrightarrow \Omega, \quad \widehat{\Omega} \text{ being the unit cube.}$$

We define the functional spaces

$$H(\mathrm{d}, \Omega) = \{ u \in L^2(\Omega) \ : \ \mathrm{d}u \in L^2(\Omega) \}, \quad \|u\|_{\mathrm{d},\Omega} = \|u\|_0 + \|\mathrm{d}u\|_0$$

$$H^1(\Omega)/\mathbb{R} \xrightarrow{\mathbf{grad}} H(\mathbf{curl}, \Omega) \xrightarrow{\mathbf{curl}} H(\mathrm{div}, \Omega) \xrightarrow{\mathrm{div}} L^2(\Omega).$$

For FEM, we have the following compatible discretization

$$\text{Nodal FE} \xrightarrow{\mathbf{grad}} \text{Edge FE} \xrightarrow{\mathbf{curl}} \text{Face FE} \xrightarrow{\mathrm{div}} \text{Disc. FE.}$$

Setting of the problem

Let Ω be a Lipschitz domain described by a NURBS mapping

$$\mathbf{F} : \widehat{\Omega} \longrightarrow \Omega, \quad \widehat{\Omega} \text{ being the unit cube.}$$

We define the functional spaces

$$H(\mathrm{d},\Omega) = \{u \in L^2(\Omega) \ : \ \mathrm{d}u \in L^2(\Omega)\}, \quad \|u\|_{\mathrm{d},\Omega} = \|u\|_0 + \|\mathrm{d}u\|_0$$

$$H^1(\Omega)/\mathbb{R} \xrightarrow{\textbf{grad}} H(\textbf{curl}\,,\Omega) \xrightarrow{\textbf{curl}} H(\mathrm{div},\Omega) \xrightarrow{\mathrm{div}} L^2(\Omega).$$

For FEM, we have the following compatible discretization

$$\text{Nodal FE} \xrightarrow{\textbf{grad}} \text{Edge FE} \xrightarrow{\textbf{curl}} \text{Face FE} \xrightarrow{\mathrm{div}} \text{Disc. FE.}$$

Approximation of vector fields: we seek for a NURBS or Splines discretization compatible with this structure.

Setting of the problem

Let Ω be a Lipschitz domain described by a NURBS mapping

$$\mathbf{F} : \widehat{\Omega} \longrightarrow \Omega, \quad \widehat{\Omega} \text{ being the unit cube.}$$

We define the functional spaces

$$H(\mathrm{d}, \Omega) = \{u \in L^2(\Omega) \ : \ \mathrm{d}u \in L^2(\Omega)\}, \quad \|u\|_{\mathrm{d},\Omega} = \|u\|_0 + \|\mathrm{d}u\|_0$$

$$H^1(\Omega)/\mathbb{R} \xrightarrow{\textbf{grad}} H(\textbf{curl}, \Omega) \xrightarrow{\textbf{curl}} H(\mathrm{div}, \Omega) \xrightarrow{\mathrm{div}} L^2(\Omega).$$

For FEM, we have the following compatible discretization

$$\text{Nodal FE} \xrightarrow{\textbf{grad}} \text{Edge FE} \xrightarrow{\textbf{curl}} \text{Face FE} \xrightarrow{\mathrm{div}} \text{Disc. FE.}$$

Approximation of vector fields: we seek for a NURBS or Splines discretization compatible with this structure.

Applications: electromagnetic problems, Darcy's flow, Stokes' flow...

Vector fields - approximation in $\widehat{\Omega}$

Buffa, Sangalli, V., 2010

To recall some **notation**: univariate B-splines and NURBS

$S^p_{\boldsymbol{\alpha}}/N^p_{\boldsymbol{\alpha}}$: univariate Splines/NURBS of degree p and regularity $\boldsymbol{\alpha}$ at knots.

$$\left\{ \frac{d}{dx} v : v \in S^p_{\boldsymbol{\alpha}} \right\} \equiv S^{p-1}_{\boldsymbol{\alpha}-1}$$

Vector fields - approximation in $\widehat{\Omega}$

Buffa, Sangalli, V., 2010

To recall some **notation**: multivariate B-splines and NURBS

$$S^{p_1,p_2,p_3}_{\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_3} = S^{p_1}_{\boldsymbol{\alpha}_1} \otimes S^{p_2}_{\boldsymbol{\alpha}_2} \otimes S^{p_3}_{\boldsymbol{\alpha}_3}, \quad N^{p_1,p_2,p_3}_{\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_3} = N^{p_1}_{\boldsymbol{\alpha}_1} \otimes N^{p_2}_{\boldsymbol{\alpha}_2} \otimes N^{p_3}_{\boldsymbol{\alpha}_3}.$$

Vector fields - approximation in $\widehat{\Omega}$

Buffa, Sangalli, V., 2010

To recall some **notation**: multivariate B-splines and NURBS

$$S^{p_1,p_2,p_3}_{\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_3} = S^{p_1}_{\boldsymbol{\alpha}_1} \otimes S^{p_2}_{\boldsymbol{\alpha}_2} \otimes S^{p_3}_{\boldsymbol{\alpha}_3}\,, \quad N^{p_1,p_2,p_3}_{\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_3} = N^{p_1}_{\boldsymbol{\alpha}_1} \otimes N^{p_2}_{\boldsymbol{\alpha}_2} \otimes N^{p_3}_{\boldsymbol{\alpha}_3}.$$

Let us not keep track of the regularity $\boldsymbol{\alpha}$ in what follows...

Vector fields - approximation in $\widehat{\Omega}$

Buffa, Sangalli, V., 2010

To recall some **notation**: multivariate B-splines and NURBS

$$S^{p_1,p_2,p_3}_{\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_3} = S^{p_1}_{\boldsymbol{\alpha}_1} \otimes S^{p_2}_{\boldsymbol{\alpha}_2} \otimes S^{p_3}_{\boldsymbol{\alpha}_3}\,, \quad N^{p_1,p_2,p_3}_{\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_3} = N^{p_1}_{\boldsymbol{\alpha}_1} \otimes N^{p_2}_{\boldsymbol{\alpha}_2} \otimes N^{p_3}_{\boldsymbol{\alpha}_3}.$$

Let us not keep track of the regularity $\boldsymbol{\alpha}$ in what follows...

Differential operators: we look for discrete spaces in $\widehat{\Omega}$ such that

$$\mathbb{R} \longrightarrow \widehat{S}_0 \xrightarrow{\widehat{\mathbf{grad}}} \widehat{S}_1 \xrightarrow{\widehat{\mathbf{curl}}} \widehat{S}_2 \xrightarrow{\widehat{\mathrm{div}}} \widehat{S}_3 \longrightarrow 0$$

Vector fields - approximation in $\widehat{\Omega}$

Buffa, Sangalli, V., 2010

To recall some **notation**: multivariate B-splines and NURBS

$$S^{p_1,p_2,p_3}_{\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_3} = S^{p_1}_{\boldsymbol{\alpha}_1} \otimes S^{p_2}_{\boldsymbol{\alpha}_2} \otimes S^{p_3}_{\boldsymbol{\alpha}_3}\,, \quad N^{p_1,p_2,p_3}_{\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_3} = N^{p_1}_{\boldsymbol{\alpha}_1} \otimes N^{p_2}_{\boldsymbol{\alpha}_2} \otimes N^{p_3}_{\boldsymbol{\alpha}_3}.$$

Let us not keep track of the regularity $\boldsymbol{\alpha}$ in what follows...

Differential operators: we look for discrete spaces in $\widehat{\Omega}$ such that

$$\mathbb{R} \longrightarrow \widehat{S}_0 \xrightarrow{\widehat{\textbf{grad}}} \widehat{S}_1 \xrightarrow{\widehat{\textbf{curl}}} \widehat{S}_2 \xrightarrow{\widehat{\mathrm{div}}} \widehat{S}_3 \longrightarrow 0$$

$$\widehat{S}_0 = S^{p_1,p_2,p_3}$$

Vector fields - approximation in $\widehat{\Omega}$

Buffa, Sangalli, V., 2010

To recall some **notation**: multivariate B-splines and NURBS

$$S^{p_1,p_2,p_3}_{\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_3} = S^{p_1}_{\boldsymbol{\alpha}_1} \otimes S^{p_2}_{\boldsymbol{\alpha}_2} \otimes S^{p_3}_{\boldsymbol{\alpha}_3}\,, \quad N^{p_1,p_2,p_3}_{\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_3} = N^{p_1}_{\boldsymbol{\alpha}_1} \otimes N^{p_2}_{\boldsymbol{\alpha}_2} \otimes N^{p_3}_{\boldsymbol{\alpha}_3}.$$

Let us not keep track of the regularity $\boldsymbol{\alpha}$ in what follows...

Differential operators: we look for discrete spaces in $\widehat{\Omega}$ such that

$$\mathbb{R} \longrightarrow \widehat{S}_0 \xrightarrow{\widehat{\mathbf{grad}}} \widehat{S}_1 \xrightarrow{\widehat{\mathbf{curl}}} \widehat{S}_2 \xrightarrow{\widehat{\mathrm{div}}} \widehat{S}_3 \longrightarrow 0$$

$$\widehat{S}_0 = S^{p_1,p_2,p_3}$$

$$\widehat{\mathbf{grad}} : \widehat{S}_0 \rightarrow S^{p_1-1,p_2,p_3} \times S^{p_1,p_2-1,p_3} \times S^{p_1,p_2,p_3-1} \equiv \widehat{S}_1$$

Vector fields - approximation in $\widehat{\Omega}$

Buffa, Sangalli, V., 2010

To recall some **notation**: multivariate B-splines and NURBS

$$S^{p_1,p_2,p_3}_{\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_3} = S^{p_1}_{\boldsymbol{\alpha}_1} \otimes S^{p_2}_{\boldsymbol{\alpha}_2} \otimes S^{p_3}_{\boldsymbol{\alpha}_3}, \quad N^{p_1,p_2,p_3}_{\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_3} = N^{p_1}_{\boldsymbol{\alpha}_1} \otimes N^{p_2}_{\boldsymbol{\alpha}_2} \otimes N^{p_3}_{\boldsymbol{\alpha}_3}.$$

Let us not keep track of the regularity $\boldsymbol{\alpha}$ in what follows...

Differential operators: we look for discrete spaces in $\widehat{\Omega}$ such that

$$\mathbb{R} \longrightarrow \widehat{S}_0 \xrightarrow{\widehat{\textbf{grad}}} \widehat{S}_1 \xrightarrow{\widehat{\textbf{curl}}} \widehat{S}_2 \xrightarrow{\widehat{\mathrm{div}}} \widehat{S}_3 \longrightarrow 0$$

$$\widehat{S}_1 = S^{p_1-1,p_2,p_3} \times S^{p_1,p_2-1,p_3} \times S^{p_1,p_2,p_3-1}$$

$$\widehat{\textbf{curl}} : \widehat{S}_1 \to S^{p_1,p_2-1,p_3-1} \times S^{p_1-1,p_2,p_3-1} \times S^{p_1-1,p_2-1,p_3} \equiv \widehat{S}_2$$

Vector fields - approximation in $\widehat{\Omega}$

Buffa, Sangalli, V., 2010

To recall some **notation**: multivariate B-splines and NURBS

$$S^{p_1,p_2,p_3}_{\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_3} = S^{p_1}_{\boldsymbol{\alpha}_1} \otimes S^{p_2}_{\boldsymbol{\alpha}_2} \otimes S^{p_3}_{\boldsymbol{\alpha}_3}\,, \quad N^{p_1,p_2,p_3}_{\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_3} = N^{p_1}_{\boldsymbol{\alpha}_1} \otimes N^{p_2}_{\boldsymbol{\alpha}_2} \otimes N^{p_3}_{\boldsymbol{\alpha}_3}.$$

Let us not keep track of the regularity $\boldsymbol{\alpha}$ in what follows...

Differential operators: we look for discrete spaces in $\widehat{\Omega}$ such that

$$\mathbb{R} \longrightarrow \widehat{S}_0 \xrightarrow{\widehat{\textbf{grad}}} \widehat{S}_1 \xrightarrow{\widehat{\textbf{curl}}} \widehat{S}_2 \xrightarrow{\widehat{\mathrm{div}}} \widehat{S}_3 \longrightarrow 0$$

$$\widehat{S}_2 = S^{p_1,p_2-1,p_3-1} \times S^{p_1-1,p_2,p_3-1} \times S^{p_1-1,p_2-1,p_3}$$

$$\widehat{\mathrm{div}} : \widehat{S}_2 \rightarrow S^{p_1-1,p_2-1,p_3-1} \equiv \widehat{S}_3$$

Vector fields - approximation in $\widehat{\Omega}$

Buffa, Sangalli, V., 2010

To recall some **notation**: multivariate B-splines and NURBS

$$S^{p_1,p_2,p_3}_{\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_3} = S^{p_1}_{\boldsymbol{\alpha}_1} \otimes S^{p_2}_{\boldsymbol{\alpha}_2} \otimes S^{p_3}_{\boldsymbol{\alpha}_3}\,, \quad N^{p_1,p_2,p_3}_{\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_3} = N^{p_1}_{\boldsymbol{\alpha}_1} \otimes N^{p_2}_{\boldsymbol{\alpha}_2} \otimes N^{p_3}_{\boldsymbol{\alpha}_3}.$$

Let us not keep track of the regularity $\boldsymbol{\alpha}$ in what follows...

Differential operators: we look for discrete spaces in $\widehat{\Omega}$ such that

$$\mathbb{R} \longrightarrow \widehat{S}_0 \xrightarrow{\widehat{\mathbf{grad}}} \widehat{S}_1 \xrightarrow{\widehat{\mathbf{curl}}} \widehat{S}_2 \xrightarrow{\widehat{\mathrm{div}}} \widehat{S}_3 \longrightarrow 0$$

- The sequence is exact.
- Generalization of nodal, edge and face elements with higher regularity.

Vector fields - approximation in $\widehat{\Omega}$

Buffa, Sangalli, V., 2010

To recall some **notation**: multivariate B-splines and NURBS

$$S^{p_1,p_2,p_3}_{\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_3} = S^{p_1}_{\boldsymbol{\alpha}_1} \otimes S^{p_2}_{\boldsymbol{\alpha}_2} \otimes S^{p_3}_{\boldsymbol{\alpha}_3}\,, \quad N^{p_1,p_2,p_3}_{\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_3} = N^{p_1}_{\boldsymbol{\alpha}_1} \otimes N^{p_2}_{\boldsymbol{\alpha}_2} \otimes N^{p_3}_{\boldsymbol{\alpha}_3}.$$

Let us not keep track of the regularity $\boldsymbol{\alpha}$ in what follows...

Differential operators: we look for discrete spaces in $\widehat{\Omega}$ such that

$$\mathbb{R} \longrightarrow \widehat{S}_0 \xrightarrow{\widehat{\mathbf{grad}}} \widehat{S}_1 \xrightarrow{\widehat{\mathbf{curl}}} \widehat{S}_2 \xrightarrow{\widehat{\mathrm{div}}} \widehat{S}_3 \longrightarrow 0$$

- The sequence is exact.
- Generalization of nodal, edge and face elements with higher regularity.

But what about NURBS?

Vector fields - approximation in $\widehat{\Omega}$

Buffa, Sangalli, V., 2010

To recall some **notation**: multivariate B-splines and NURBS

$$S^{p_1,p_2,p_3}_{\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_3} = S^{p_1}_{\boldsymbol{\alpha}_1} \otimes S^{p_2}_{\boldsymbol{\alpha}_2} \otimes S^{p_3}_{\boldsymbol{\alpha}_3}\,, \quad N^{p_1,p_2,p_3}_{\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_3} = N^{p_1}_{\boldsymbol{\alpha}_1} \otimes N^{p_2}_{\boldsymbol{\alpha}_2} \otimes N^{p_3}_{\boldsymbol{\alpha}_3}.$$

Let us not keep track of the regularity $\boldsymbol{\alpha}$ in what follows...

Differential operators: we look for discrete spaces in $\widehat{\Omega}$ such that

$$\widehat{N}_0 \xrightarrow{\cancel{\widehat{\mathbf{grad}}}} \widehat{N}_1 \xrightarrow{\cancel{\widehat{\mathbf{curl}}}} \widehat{N}_2 \xrightarrow{\cancel{\widehat{\mathrm{div}}}} \widehat{N}_3$$

The derivative of a NURBS is not a NURBS.

The diagram for NURBS does not hold true.

D. Arnold (et al.) 06-10 teach us that $\widehat{N}_i$ are then not suitable for vector fields approximations

Vector fields - SPLINES approximation in Ω

Spaces on Ω are defined just by **push forward**:

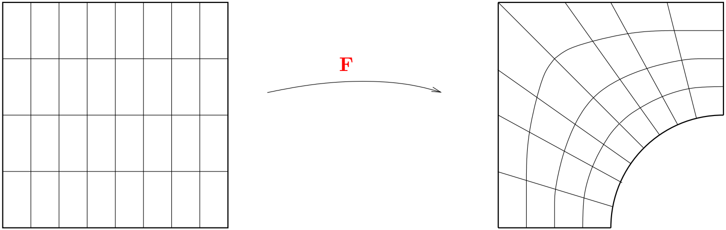
$$S_i = \iota^i(\widehat{S}_i), \qquad N_i = \iota^i(\widehat{N}_i).$$

Vector fields - SPLINES approximation in Ω

Spaces on Ω are defined just by **push forward**:

$$S_i = \iota^i(\widehat{S}_i), \qquad N_i = \iota^i(\widehat{N}_i).$$

$$\iota^0(\widehat{\phi}) \circ \mathbf{F} = \widehat{\phi}, \qquad \text{standard mapping}$$

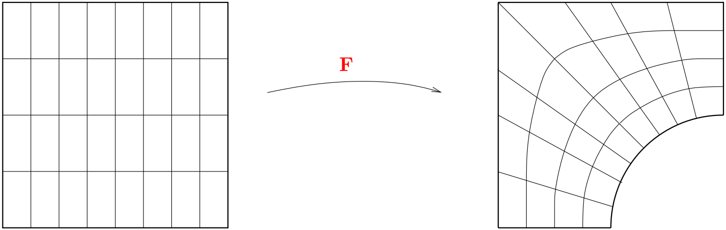


Vector fields - SPLINES approximation in Ω

Spaces on Ω are defined just by **push forward**:

$$S_i = \iota^i(\widehat{S}_i), \qquad N_i = \iota^i(\widehat{N}_i).$$

$$\iota^1(\widehat{\mathbf{u}}) \circ \mathbf{F} = D\mathbf{F}^{-T}\widehat{\mathbf{u}}, \qquad \text{curl conforming mapping}$$

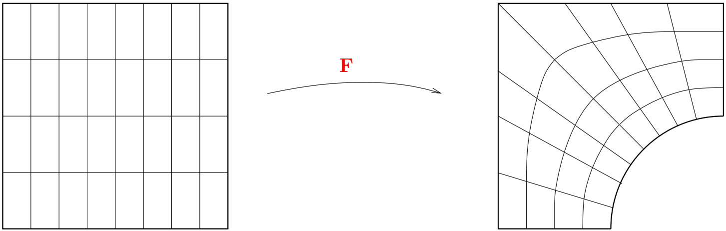


Vector fields - SPLINES approximation in Ω

Spaces on Ω are defined just by **push forward**:

$$S_i = \iota^i(\widehat{S}_i), \qquad N_i = \iota^i(\widehat{N}_i).$$

$$\iota^2(\widehat{\mathbf{u}}) \circ \mathbf{F} = (\det D\mathbf{F})^{-1} D\mathbf{F}\,\widehat{\mathbf{u}}, \qquad \text{Piola mapping}$$

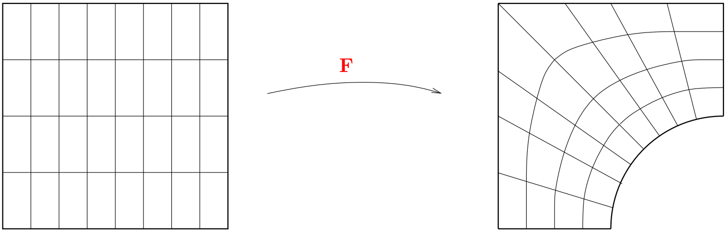


Vector fields - SPLINES approximation in Ω

Spaces on Ω are defined just by **push forward**:

$$S_i = \iota^i(\widehat{S}_i), \qquad N_i = \iota^i(\widehat{N}_i).$$

$$\iota^3(\phi) \circ \mathbf{F} = (\det D\mathbf{F})^{-1}\widehat{\phi}, \qquad \text{3-forms mapping}$$



Vector fields - SPLINES approximation in Ω

Spaces on Ω are defined just by **push forward**:

$$S_i = \iota^i(\widehat{S}_i), \qquad N_i = \iota^i(\widehat{N}_i).$$

For splines, everything then works thanks to the **geometric structure**:

$$H^1(\Omega)/\mathbb{R} \xrightarrow{\textbf{grad}} H(\textbf{curl}\,,\Omega) \xrightarrow{\textbf{curl}} H(\mathrm{div},\Omega) \xrightarrow{\mathrm{div}} L^2(\Omega)$$

$$S_0/\mathbb{R} \xrightarrow{\textbf{grad}} S_1 \xrightarrow{\textbf{curl}} S_2 \xrightarrow{\mathrm{div}} S_3$$

Vector fields - SPLINES approximation in Ω

Spaces on Ω are defined just by **push forward**:

$$S_i = \iota^i(\widehat{S}_i), \qquad N_i = \iota^i(\widehat{N}_i).$$

For splines, everything then works thanks to the **geometric structure**:

$$\begin{array}{ccccccc}
H^1(\Omega)/\mathbb{R} & \xrightarrow{\textbf{grad}} & H(\textbf{curl}\,,\Omega) & \xrightarrow{\textbf{curl}} & H(\mathrm{div},\Omega) & \xrightarrow{\mathrm{div}} & L^2(\Omega) \\
\Pi^0?\big\downarrow & & \Pi^1?\big\downarrow & & \Pi^2?\big\downarrow & & \Pi^3?\big\downarrow \\
S_0/\mathbb{R} & \xrightarrow{\textbf{grad}} & S_1 & \xrightarrow{\textbf{curl}} & S_2 & \xrightarrow{\mathrm{div}} & S_3
\end{array}$$

How can we define the commuting interpolants Π^i?

Vector fields - SPLINES approximation in Ω

Spaces on Ω are defined just by **push forward**:

$$S_i = \iota^i(\widehat{S}_i), \qquad N_i = \iota^i(\widehat{N}_i).$$

For splines, everything then works thanks to the **geometric structure**:

$$\begin{array}{ccccccc}
H^1(\Omega)/\mathbb{R} & \xrightarrow{\textbf{grad}} & H(\textbf{curl}\,,\Omega) & \xrightarrow{\textbf{curl}} & H(\mathrm{div},\Omega) & \xrightarrow{\mathrm{div}} & L^2(\Omega) \\
\Pi^0 ? \big\downarrow & & \Pi^1 ? \big\downarrow & & \Pi^2 ? \big\downarrow & & \Pi^3 ? \big\downarrow \\
S_0/\mathbb{R} & \xrightarrow{\textbf{grad}} & S_1 & \xrightarrow{\textbf{curl}} & S_2 & \xrightarrow{\mathrm{div}} & S_3
\end{array}$$

How can we define the commuting interpolants Π^i?

- Construct (commuting) one-dimensional projectors.
- Construct the projectors in $\widehat{\Omega}$ with tensor product structure.
- Pull back from Ω to $\widehat{\Omega}$.

Construction of the interpolants for Spline spaces

Buffa, Rivas, Sangalli, V., 2010

Starting point: we know the **local** and **stable** projector $\widehat{\Pi}_S^p$.

$$\widehat{\Pi}_S^p s = s, \quad \forall s \in S_{\boldsymbol{\alpha}}^p,$$
$$|u|_{H^k(I)} \leq C|u|_{H^k(\widetilde{I})}, \quad \forall u \in H^k(0,1),\ 0 \leq k \leq p+1,$$

where $I = (\xi_j, \xi_{j+1})$, and $\widetilde{I}$ is a local extension.

L.L. Schumaker. Spline functions: basic theory, 2007.

Y. Bazilevs, L. Beirão da Veiga, J. Cottrell, T.J.R. Hughes, G. Sangalli, 2006.

Construction of the interpolants for Spline spaces

Buffa, Rivas, Sangalli, V., 2010

Starting point: we know the **local** and **stable** projector $\widehat{\Pi}_S^p$.

First idea: tensor products of $\widehat{\Pi}_S^p$, choosing the right degree p.

Construction of the interpolants for Spline spaces

Buffa, Rivas, Sangalli, V., 2010

Starting point: we know the **local** and **stable** projector $\widehat{\Pi}_S^p$.

First idea: tensor products of $\widehat{\Pi}_S^p$, choosing the right degree p.

The diagram is **not commutative**, because even the 1D diagram is not:

$$\widehat{\Pi}_S^{p-1} \frac{d}{dx} u \neq \frac{d}{dx} \widehat{\Pi}_S^p u.$$

Construction of the interpolants for Spline spaces

Buffa, Rivas, Sangalli, V., 2010

Starting point: we know the **local** and **stable** projector $\widehat{\Pi}_S^p$.

Second idea: define $\widehat{\Pi}_A^{p-1}$ such that the 1D diagram commutes.

$$
\begin{tikzcd}
\mathbb{R} \arrow[r] & H^1(0,1) \arrow[r, "\frac{d}{dx}"] \arrow[d, "\widehat{\Pi}_S^p"'] & L^2(0,1) \arrow[r] \arrow[d, "\widehat{\Pi}_A^{p-1}"'] & 0 \\
\mathbb{R} \arrow[r] & S_{\boldsymbol{\alpha}}^p \arrow[r, "\frac{d}{dx}"] & S_{\boldsymbol{\alpha}-1}^{p-1} \arrow[r] & 0
\end{tikzcd}
$$

Construction of the interpolants for Spline spaces

Buffa, Rivas, Sangalli, V., 2010

Starting point: we know the **local** and **stable** projector $\widehat{\Pi}_S^p$.

Second idea: define $\widehat{\Pi}_A^{p-1}$ such that the 1D diagram commutes.

$$
\begin{tikzcd}
\mathbb{R} \arrow[r] & H^1(0,1) \arrow[r, "\frac{d}{dx}"] \arrow[d, "\widehat{\Pi}_S^p"'] & L^2(0,1) \arrow[r] \arrow[d, "\widehat{\Pi}_A^{p-1}"'] & 0 \\
\mathbb{R} \arrow[r] & S_{\boldsymbol{\alpha}}^p \arrow[r, "\frac{d}{dx}"] & S_{\boldsymbol{\alpha}-1}^{p-1} \arrow[r] & 0
\end{tikzcd}
$$

So, we define it as

$$\widehat{\Pi}_A^{p-1} v := \frac{d}{dx} \widehat{\Pi}_S^p \int_0^x v(\xi) d\xi, \quad \forall v \in L^2(0,1).$$

Construction of the interpolants for Spline spaces

Buffa, Rivas, Sangalli, V., 2010

Starting point: we know the **local** and **stable** projector $\widehat{\Pi}_S^p$.

Second idea: define $\widehat{\Pi}_A^{p-1}$ such that the 1D diagram commutes.

$$
\begin{tikzcd}
\mathbb{R} \arrow[r] & H^1(0,1) \arrow[r, "\frac{d}{dx}"] \arrow[d, "\widehat{\Pi}_S^p"'] & L^2(0,1) \arrow[r] \arrow[d, "\widehat{\Pi}_A^{p-1}"'] & 0 \\
\mathbb{R} \arrow[r] & S_{\boldsymbol{\alpha}}^p \arrow[r, "\frac{d}{dx}"] & S_{\boldsymbol{\alpha}-1}^{p-1} \arrow[r] & 0
\end{tikzcd}
$$

So, we define it as

$$\widehat{\Pi}_A^{p-1} v := \frac{d}{dx} \widehat{\Pi}_S^p \int_0^x v(\xi) d\xi, \quad \forall v \in L^2(0,1).$$

It is a **stable** and **local** projector:

$$\widehat{\Pi}_A^{p-1} s = s, \quad \forall s \in S_{\boldsymbol{\alpha}-1}^{p-1},$$

$$|u|_{H^k(I)} \leq C |u|_{H^k(\widetilde{I})}, \quad \forall u \in H^k(0,1), \, 0 \leq k \leq p.$$

Construction of the interpolants for Spline spaces

Buffa, Rivas, Sangalli, V., 2010

Starting point: we know the **local** and **stable** projector $\widehat{\Pi}_S^p$.

Second idea: define $\widehat{\Pi}_A^{p-1}$ such that the 1D diagram commutes.

Next step: in $\widehat{\Omega}$, take tensor products of $\widehat{\Pi}_S^p$ and $\widehat{\Pi}_A^{p-1}$.

For instance, for

$$\widehat{S}^1 = S^{p_1-1,p_2,p_3} \times S^{p_1,p_2-1,p_3} \times S^{p_1,p_2,p_3-1}$$

we define

$$\widehat{\Pi}^1 := (\widehat{\Pi}_A^{p_1-1} \otimes \widehat{\Pi}_S^{p_2} \otimes \widehat{\Pi}_S^{p_3}) \times (\widehat{\Pi}_S^{p_1} \otimes \widehat{\Pi}_A^{p_2-1} \otimes \widehat{\Pi}_S^{p_3}) \times (\widehat{\Pi}_S^{p_1} \otimes \widehat{\Pi}_S^{p_2} \otimes \widehat{\Pi}_A^{p_3-1}).$$

Construction of the interpolants for Spline spaces

Buffa, Rivas, Sangalli, V., 2010

Starting point: we know the **local** and **stable** projector $\widehat{\Pi}_S^p$.

Second idea: define $\widehat{\Pi}_A^{p-1}$ such that the 1D diagram commutes.

Next step: in $\widehat{\Omega}$, take tensor products of $\widehat{\Pi}_S^p$ and $\widehat{\Pi}_A^{p-1}$.

With this choice, the **diagram** in $\widehat{\Omega}$ is **commutative**.

$$\begin{array}{ccccccc}
H^1(\widehat{\Omega})/\mathbb{R} & \xrightarrow{\widehat{\mathbf{grad}}} & H(\mathbf{curl}\,,\widehat{\Omega}) & \xrightarrow{\widehat{\mathbf{curl}}} & H(\mathrm{div},\widehat{\Omega}) & \xrightarrow{\widehat{\mathrm{div}}} & L^2(\widehat{\Omega}) \\
\widehat{\Pi}^0 \Big\downarrow & & \widehat{\Pi}^1 \Big\downarrow & & \widehat{\Pi}^2 \Big\downarrow & & \widehat{\Pi}^3 \Big\downarrow \\
\widehat{S}_0/\mathbb{R} & \xrightarrow{\widehat{\mathbf{grad}}} & \widehat{S}_1 & \xrightarrow{\widehat{\mathbf{curl}}} & \widehat{S}_2 & \xrightarrow{\widehat{\mathrm{div}}} & \widehat{S}_3
\end{array}$$

Construction of the interpolants for Spline spaces

Buffa, Rivas, Sangalli, V., 2010

Starting point: we know the **local** and **stable** projector $\widehat{\Pi}_S^p$.

Second idea: define $\widehat{\Pi}_A^{p-1}$ such that the 1D diagram commutes.

Next step: in $\widehat{\Omega}$, take tensor products of $\widehat{\Pi}_S^p$ and $\widehat{\Pi}_A^{p-1}$.

Last step: define the interpolants in Ω by **push forward** techniques.

$$\iota^i(\Pi^i \phi) = \widehat{\Pi}^i(\iota^i(\phi)), \quad \forall \phi \in H(\mathrm{d}, \Omega), \quad i = 0, \ldots, 3$$

Construction of the interpolants for Spline spaces

Buffa, Rivas, Sangalli, V., 2010

Starting point: we know the **local** and **stable** projector $\widehat{\Pi}_S^p$.

Second idea: define $\widehat{\Pi}_A^{p-1}$ such that the 1D diagram commutes.

Next step: in $\widehat{\Omega}$, take tensor products of $\widehat{\Pi}_S^p$ and $\widehat{\Pi}_A^{p-1}$.

Last step: define the interpolants in Ω by **push forward** techniques.

With the definition of these interpolants, **the diagram is commutative**.

$$
\begin{array}{ccccccc}
H^1(\Omega)/\mathbb{R} & \xrightarrow{\textbf{grad}} & H(\textbf{curl}\,,\Omega) & \xrightarrow{\textbf{curl}} & H(\mathrm{div},\Omega) & \xrightarrow{\mathrm{div}} & L^2(\Omega) \\
\Pi^0 \big\downarrow & & \Pi^1 \big\downarrow & & \Pi^2 \big\downarrow & & \Pi^3 \big\downarrow \\
S_0/\mathbb{R} & \xrightarrow{\textbf{grad}} & S_1 & \xrightarrow{\textbf{curl}} & S_2 & \xrightarrow{\mathrm{div}} & S_3
\end{array}
$$

Construction of the interpolants for Spline spaces

Buffa, Rivas, Sangalli, V., 2010

Starting point: we know the **local** and **stable** projector $\widehat{\Pi}_S^p$.

Second idea: define $\widehat{\Pi}_A^{p-1}$ such that the 1D diagram commutes.

Next step: in $\widehat{\Omega}$, take tensor products of $\widehat{\Pi}_S^p$ and $\widehat{\Pi}_A^{p-1}$.

Last step: define the interpolants in Ω by **push forward** techniques.

With the definition of these interpolants, **the diagram is commutative**.

$$\begin{array}{ccccccc}
H^1(\Omega)/\mathbb{R} & \xrightarrow{\textbf{grad}} & H(\textbf{curl}\,,\Omega) & \xrightarrow{\textbf{curl}} & H(\mathrm{div},\Omega) & \xrightarrow{\mathrm{div}} & L^2(\Omega) \\
\Pi^0 \big\downarrow & & \Pi^1 \big\downarrow & & \Pi^2 \big\downarrow & & \Pi^3 \big\downarrow \\
S_0/\mathbb{R} & \xrightarrow{\textbf{grad}} & S_1 & \xrightarrow{\textbf{curl}} & S_2 & \xrightarrow{\mathrm{div}} & S_3
\end{array}$$

The projectors Π^i are **local** and $L^2(\Omega)$-**stable**.

Construction of the interpolants for Spline spaces

Buffa, Rivas, Sangalli, V., 2010

Starting point: we know the **local** and **stable** projector $\widehat{\Pi}_S^p$.

Second idea: define $\widehat{\Pi}_A^{p-1}$ such that the 1D diagram commutes.

Next step: in $\widehat{\Omega}$, take tensor products of $\widehat{\Pi}_S^p$ and $\widehat{\Pi}_A^{p-1}$.

Last step: define the interpolants in Ω by **push forward** techniques.

With the definition of these interpolants, **the diagram is commutative**.

$$
\begin{tikzcd}
H^1(\Omega)/\mathbb{R} \arrow[r, "\mathbf{grad}"] \arrow[d, "\Pi^0"'] & H(\mathbf{curl}, \Omega) \arrow[r, "\mathbf{curl}"] \arrow[d, "\Pi^1"'] & H(\mathrm{div}, \Omega) \arrow[r, "\mathrm{div}"] \arrow[d, "\Pi^2"'] & L^2(\Omega) \arrow[d, "\Pi^3"'] \\
S_0/\mathbb{R} \arrow[r, "\mathbf{grad}"] & S_1 \arrow[r, "\mathbf{curl}"] & S_2 \arrow[r, "\mathrm{div}"] & S_3
\end{tikzcd}
$$

The projectors Π^i are **local** and $L^2(\Omega)$-**stable**.

The same idea can be used in spaces with boundary conditions.

Approximation estimates

Buffa, Rivas, Sangalli, V., 2010.

Assume $\mathbf{F} : \widehat{\Omega} \to \Omega$ belongs to N_0 and $\mathbf{F}^{-1}$ is piecewise regular.

- Let K be an element of the mesh on the physical domain: $0 \leq \ell \leq p$

$$\|u - \Pi^i u\|_{H(d^i,K)} \leq C h^\ell \|u\|_{H^\ell(d^i,\widetilde{K})}, \qquad i = 0, \ldots, 3,$$

Approximation estimates

Buffa, Rivas, Sangalli, V., 2010.

Assume $\mathbf{F} : \widehat{\Omega} \to \Omega$ belongs to N_0 and $\mathbf{F}^{-1}$ is piecewise regular.

- Let K be an element of the mesh on the physical domain: $0 \leq \ell \leq p$

$$\|u - \Pi^i u\|_{H(d^i,K)} \leq Ch^\ell \|u\|_{H^\ell(d^i,\widetilde{K})}, \qquad i = 0, \dots, 3,$$

As a consequence

$$\|\mathbf{u} - \Pi^1 \mathbf{u}\|_{H(\mathbf{curl},\Omega)} \leq Ch^\ell \|\mathbf{u}\|_{H^\ell(\mathbf{curl},\Omega)}.$$

Approximation estimates

Buffa, Rivas, Sangalli, V., 2010.

Assume $\mathbf{F} : \widehat{\Omega} \to \Omega$ belongs to N_0 and $\mathbf{F}^{-1}$ is piecewise regular.

- Let K be an element of the mesh on the physical domain: $0 \leq \ell \leq p$

$$\|u - \Pi^i u\|_{H(d^i,K)} \leq Ch^\ell \|u\|_{H^\ell(d^i,\widetilde{K})}, \qquad i = 0, \dots, 3,$$

As a consequence

$$\|\mathbf{u} - \Pi^1 \mathbf{u}\|_{H(\mathbf{curl},\Omega)} \leq Ch^\ell \|\mathbf{u}\|_{H^\ell(\mathbf{curl},\Omega)}.$$

Reducing the continuity we can obtain estimates in terms of p.

- If $\alpha = \max\{\boldsymbol{\alpha}_i\} \leq \frac{p-1}{2}$ then

$$\|u - \Pi^i u\|_{H(d^i,K)} \leq C(h/p)^\ell \|u\|_{H^\ell(d^i,\widetilde{K})}, \qquad i = 0, \dots, 3.$$

Beirão da Veiga, Buffa, Rivas, Sangalli (2009).

Maxwell eigenproblem: B-splines discretization

Maxwell eigenproblem

Find $\mathbf{u} \in H_0(\mathbf{curl}\,, \Omega)$, $\mathbf{u} \neq \mathbf{0}$ and $\lambda \in \mathbb{R}$, $\lambda \neq 0$ such that

$$\int_\Omega \mathbf{curl}\,\mathbf{u} \cdot \mathbf{curl}\,\mathbf{v} = \lambda \int_\Omega \mathbf{u} \cdot \mathbf{v} \qquad \forall \mathbf{v} \in H_0(\mathbf{curl}\,, \Omega)$$

Maxwell eigenproblem: B-splines discretization

Maxwell eigenproblem

Find $\mathbf{u} \in H_0(\mathbf{curl}\,, \Omega)$, $\mathbf{u} \neq \mathbf{0}$ and $\lambda \in \mathbb{R}$, $\lambda \neq 0$ such that

$$\int_\Omega \mathbf{curl}\,\mathbf{u} \cdot \mathbf{curl}\,\mathbf{v} = \lambda \int_\Omega \mathbf{u} \cdot \mathbf{v} \qquad \forall \mathbf{v} \in H_0(\mathbf{curl}\,, \Omega)$$

Theorem

If there is a commuting diagram with **local** and $L^2(\Omega)$-**stable** projectors, the Galerkin approximation is **spurious-free** and **optimal**.

D. Arnold, R. Falk, R. Winther, Bulletin A.M.S. (2010)

Maxwell eigenproblem: B-splines discretization

Maxwell eigenproblem

Find $\mathbf{u} \in H_0(\mathbf{curl}\,, \Omega)$, $\mathbf{u} \neq \mathbf{0}$ and $\lambda \in \mathbb{R}$, $\lambda \neq 0$ such that

$$\int_\Omega \mathbf{curl}\,\mathbf{u} \cdot \mathbf{curl}\,\mathbf{v} = \lambda \int_\Omega \mathbf{u} \cdot \mathbf{v} \qquad \forall \mathbf{v} \in H_0(\mathbf{curl}\,, \Omega)$$

Theorem

If there is a commuting diagram with **local** and $L^2(\Omega)$-**stable** projectors, the Galerkin approximation is **spurious-free** and **optimal**.

D. Arnold, R. Falk, R. Winther, Bulletin A.M.S. (2010)

The B-spline discretization fulfills the theorem: it is **spectrally correct**.

- Exact CAD description of the geometry.
- Higher regularity than standard edge elements.
- Singular functions are approximated correctly.

Maxwell eigenproblem in the patch $\widehat{\Omega}$

We solve the eigenvalue problem: *Find* $\mathbf{u} \neq 0$ *and* $\omega \neq 0$ *such that*

$$\begin{aligned} \mathbf{curl}\,\mathbf{curl}\,\mathbf{u} = \omega^2 \mathbf{u} \quad & \text{in } \widehat{\Omega}, \\ \mathbf{u} \times \mathbf{n} = \mathbf{0} \quad & \text{on } \partial\widehat{\Omega}. \end{aligned}$$

We solve with a Galerkin projection on $\widehat{S}_{1,0}$, and different degrees p.

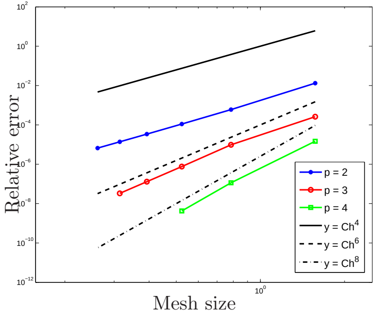
Maxwell eigenproblem in the patch $\widehat{\Omega}$

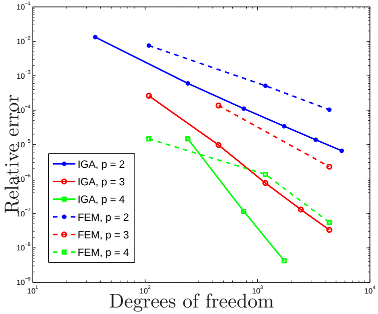
We solve the eigenvalue problem: *Find* $\mathbf{u} \neq 0$ *and* $\omega \neq 0$ *such that*

$$\begin{aligned} \mathbf{curl}\,\mathbf{curl}\,\mathbf{u} = \omega^2 \mathbf{u} \quad & \text{in } \widehat{\Omega}, \\ \mathbf{u} \times \mathbf{n} = \mathbf{0} \quad & \text{on } \partial\widehat{\Omega}. \end{aligned}$$

We solve with a Galerkin projection on $\widehat{S}_{1,0}$, and different degrees p.

In terms of d.o.f., we obtain better convergence rate than edge elements.



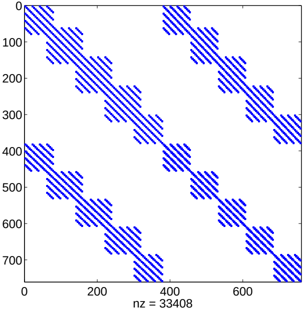


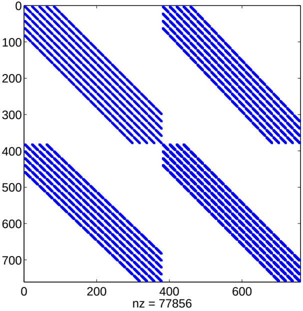
Maxwell eigenproblem in the patch $\widehat{\Omega}$

We solve the eigenvalue problem: *Find* $\mathbf{u} \neq 0$ *and* $\omega \neq 0$ *such that*

$$\begin{aligned} \mathbf{curl}\,\mathbf{curl}\,\mathbf{u} = \omega^2 \mathbf{u} \quad & \text{in } \widehat{\Omega}, \\ \mathbf{u} \times \mathbf{n} = \mathbf{0} \quad & \text{on } \partial\widehat{\Omega}. \end{aligned}$$

Fill-in of the matrix with respect to FEM. Sparsity pattern is similar.





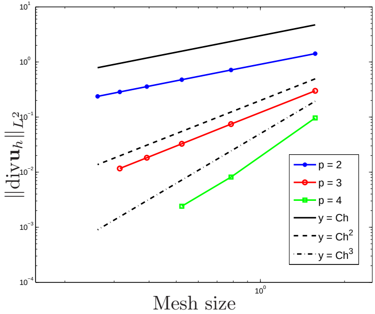
Maxwell eigenproblem in the patch $\widehat{\Omega}$

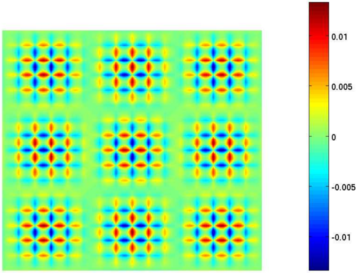
We solve the eigenvalue problem: *Find* $\mathbf{u} \neq 0$ *and* $\omega \neq 0$ *such that*

$$\begin{aligned} \mathbf{curl}\,\mathbf{curl}\,\mathbf{u} = \omega^2 \mathbf{u} \quad & \text{in } \widehat{\Omega}, \\ \mathbf{u} \times \mathbf{n} = \mathbf{0} \quad & \text{on } \partial\widehat{\Omega}. \end{aligned}$$

We solve with continuous fields: the **divergence** is well defined.

It is an oscillating field, and converges to zero with order h^{p-1}.



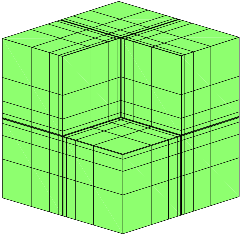


Maxwell eigenproblem: Fichera's corner

We solve the eigenvalue problem: *Find* $\mathbf{u} \neq 0$ *and* $\omega \neq 0$ *such that*

$$\begin{aligned} \mathbf{curl}\,\mathbf{curl}\,\mathbf{u} &= \omega^2 \mathbf{u} \quad \text{in } \Omega, \\ \mathbf{u} \times \mathbf{n} &= \mathbf{0} \quad \text{on } \partial\Omega. \end{aligned}$$

Interaction between edge and corner singularities.



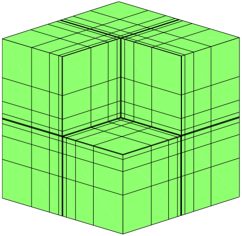
Maxwell eigenproblem: Fichera's corner

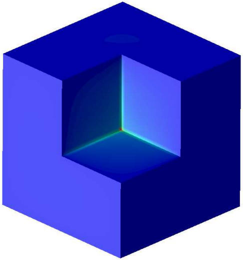
We solve the eigenvalue problem: *Find* $\mathbf{u} \neq 0$ *and* $\omega \neq 0$ *such that*

$$\begin{aligned} \mathbf{curl}\,\mathbf{curl}\,\mathbf{u} &= \omega^2 \mathbf{u} \quad \text{in } \Omega, \\ \mathbf{u} \times \mathbf{n} &= \mathbf{0} \quad \text{on } \partial\Omega. \end{aligned}$$

Interaction between edge and corner singularities.

We get a good approximation of the singular functions.





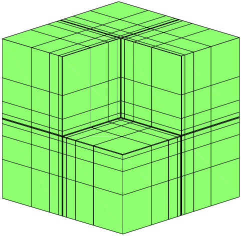
Maxwell eigenproblem: Fichera's corner

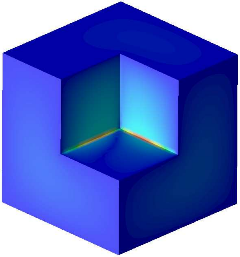
We solve the eigenvalue problem: *Find* $\mathbf{u} \neq 0$ *and* $\omega \neq 0$ *such that*

$$\begin{aligned} \mathbf{curl}\,\mathbf{curl}\,\mathbf{u} &= \omega^2 \mathbf{u} \quad \text{in } \Omega, \\ \mathbf{u} \times \mathbf{n} &= \mathbf{0} \quad \text{on } \partial\Omega. \end{aligned}$$

Interaction between edge and corner singularities.

We get a good approximation of the singular functions.





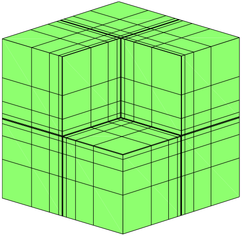
Maxwell eigenproblem: Fichera's corner

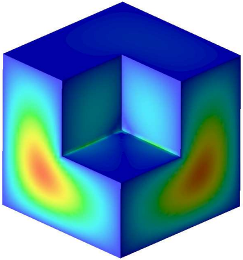
We solve the eigenvalue problem: *Find* $\mathbf{u} \neq 0$ *and* $\omega \neq 0$ *such that*

$$\begin{aligned} \mathbf{curl}\,\mathbf{curl}\,\mathbf{u} = \omega^2 \mathbf{u} &\quad \text{in } \Omega, \\ \mathbf{u} \times \mathbf{n} = \mathbf{0} &\quad \text{on } \partial\Omega. \end{aligned}$$

Interaction between edge and corner singularities.

We get a good approximation of the singular functions.





Maxwell eigenproblem: Fichera's corner

We solve the eigenvalue problem: *Find* $\mathbf{u} \neq 0$ *and* $\omega \neq 0$ *such that*

$$\begin{aligned} \mathbf{curl}\,\mathbf{curl}\,\mathbf{u} = \omega^2 \mathbf{u} \quad & \text{in } \Omega, \\ \mathbf{u} \times \mathbf{n} = \mathbf{0} \quad & \text{on } \partial\Omega. \end{aligned}$$

Interaction between edge and corner singularities.

We get a good approximation of the singular functions.

	Eigenvalues computation		
CODE by	S. Zaglmayr	M. Duruflé	IGA, p = 3
d.o.f.	53982	177720	8421
Eig. 1.	3.2199939	3.2198740	3.2194306
Eig. 2.	5.8804425	5.88041891	5.8804604
Eig. 3.	5.8804553	5.88041891	5.8804604
Eig. 4.	10.6856632	10.6854921	10.6866214
Eig. 5.	10.6936955	10.6937829	10.6949643
Eig. 6.	10.6937289	10.6937829	10.6949643
Eig. 7.	12.3168796	12.3165205	12.3179492
Eig. 8.	12.3176901	12.3165205	12.3179492

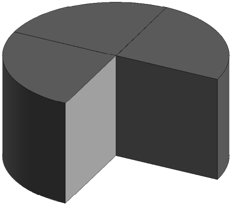
Maxwell source problem: non-convex domain

We solve the source problem, with mixed boundary conditions.

$$\mathbf{curl}\,\mathbf{curl}\,\mathbf{u} + \mathbf{u} = \mathbf{f}.$$

The geometry is described **exactly** with only three elements.

Domain with a reentrant edge (as the L-shaped domain).



Maxwell source problem: non-convex domain

We solve the source problem, with mixed boundary conditions.

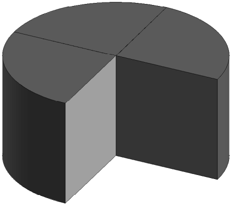
$$\mathbf{curl}\,\mathbf{curl}\,\mathbf{u} + \mathbf{u} = \mathbf{f}.$$

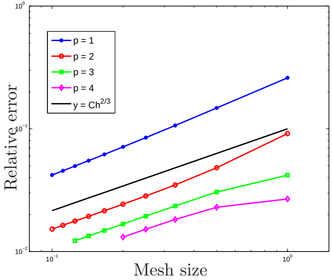
The geometry is described **exactly** with only three elements.

Domain with a reentrant edge (as the L-shaped domain).

The solution is $\mathbf{u} = \mathbf{grad}(r^{2/3}\sin(2\theta/3))$, and $\mathbf{u} \in H^{2/3-\varepsilon}(\mathbf{curl}\,,\Omega)$.

The convergence rate in energy norm is $h^{2/3}$, as expected.





Maxwell eigenproblem: NURBS discretization

If we try to discretize the problem with NURBS, in the form:
Find $\mathbf{u}_h \in N_1$, $\mathbf{u}_h \neq \mathbf{0}$ *and* $\lambda \in \mathbb{R}$, $\lambda \neq 0$ *such that*

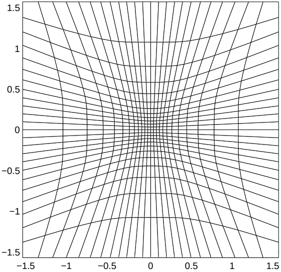
$$\int_\Omega \mathbf{curl}\,\mathbf{u}_h \cdot \mathbf{curl}\,\mathbf{v}_h = \lambda \int_\Omega \mathbf{u}_h \cdot \mathbf{v}_h \quad \forall \mathbf{v}_h \in N_1,$$

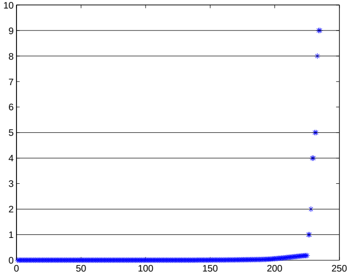
Maxwell eigenproblem: NURBS discretization

If we try to discretize the problem with NURBS, in the form:
Find $\mathbf{u}_h \in N_1$, $\mathbf{u}_h \neq \mathbf{0}$ *and* $\lambda \in \mathbb{R}$, $\lambda \neq 0$ *such that*

$$\int_\Omega \mathbf{curl}\,\mathbf{u}_h \cdot \mathbf{curl}\,\mathbf{v}_h = \lambda \int_\Omega \mathbf{u}_h \cdot \mathbf{v}_h \quad \forall \mathbf{v}_h \in N_1,$$

the diagram does not hold: **spurious eigenvalues** appear.





Maxwell eigenproblem: NURBS discretization

If we try to discretize the problem with NURBS, in the form:
Find $\mathbf{u}_h \in N_1$, $\mathbf{u}_h \neq \mathbf{0}$ *and* $\lambda \in \mathbb{R}$, $\lambda \neq 0$ *such that*

$$\int_\Omega \mathbf{curl}\, \mathbf{u}_h \cdot \mathbf{curl}\, \mathbf{v}_h = \lambda \int_\Omega \mathbf{u}_h \cdot \mathbf{v}_h \quad \forall \mathbf{v}_h \in N_1,$$

the diagram does not hold: **spurious eigenvalues** appear.

In the continuous case, we have the equivalent **mixed formulation**:

Find $(\mathbf{u}, p) \in H_0(\mathbf{curl}\,, \Omega) \times H_0^1(\Omega)$, $\lambda \in \mathbb{R}$ *such that*

$$\begin{aligned} \int_\Omega \mathbf{curl}\, \mathbf{u} \cdot \mathbf{curl}\, \mathbf{v} + \int_\Omega \nabla p \cdot \mathbf{v} &= \lambda \int_\Omega \mathbf{u} \cdot \mathbf{v} \quad && \forall \mathbf{v} \in H_0(\mathbf{curl}\,, \Omega), \\ \int_\Omega \nabla q \cdot \mathbf{u} &= 0 && \forall q \in H_0^1(\Omega). \end{aligned}$$

The two discrete formulations are equivalent for B-splines.

Maxwell eigenproblem: NURBS discretization

If we try to discretize the problem with NURBS, in the form:
Find $\mathbf{u}_h \in N_1$, $\mathbf{u}_h \neq \mathbf{0}$ *and* $\lambda \in \mathbb{R}$, $\lambda \neq 0$ *such that*

$$\int_\Omega \mathbf{curl}\,\mathbf{u}_h \cdot \mathbf{curl}\,\mathbf{v}_h = \lambda \int_\Omega \mathbf{u}_h \cdot \mathbf{v}_h \quad \forall \mathbf{v}_h \in N_1,$$

the diagram does not hold: **spurious eigenvalues** appear.

For a **NURBS** discretization, the two formulations are **not equivalent**.
Find $(\mathbf{u}_h, p_h) \in N_1 \times N_0$, $\lambda \in \mathbb{R}$ *such that*

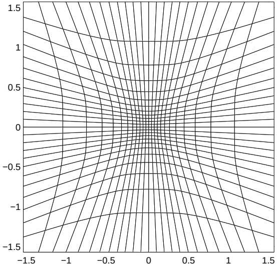
$$\begin{aligned} \int_\Omega \mathbf{curl}\,\mathbf{u}_h \cdot \mathbf{curl}\,\mathbf{v}_h + \int_\Omega \nabla p_h \cdot \mathbf{v}_h &= \lambda \int_\Omega \mathbf{u}_h \cdot \mathbf{v}_h \quad && \forall \mathbf{v}_h \in N_1, \\ \int_\Omega \nabla q_h \cdot \mathbf{u}_h &= 0 && \forall q_h \in N_0. \end{aligned}$$

NURBS discretization: numerical results

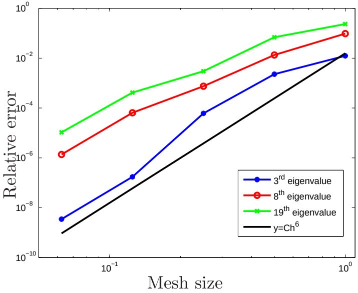
Buffa, Sangalli, V. In preparation

The domain Ω is described with a NURBS mapping.

Solution of the mixed formulation with a NURBS discretization.



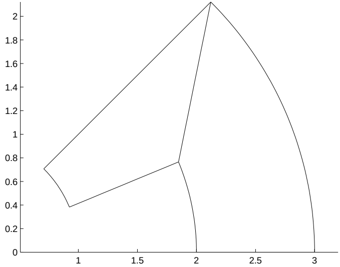
NURBS map of fine mesh



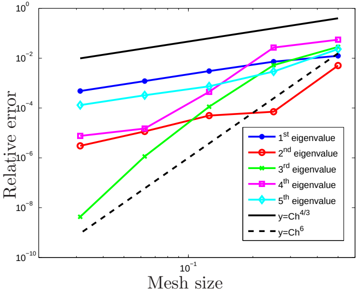
- The results are **spurious-free**, with optimal convergence rate.

The domain Ω is described with a NURBS mapping.

Solution of the mixed formulation with a NURBS discretization.



NURBS map of the coarsest mesh



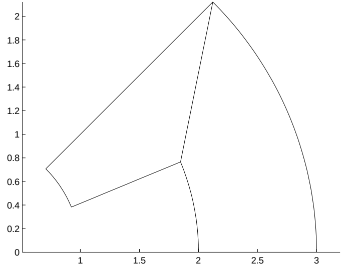
- The results are **spurious-free**, with optimal convergence rate.
- Also the singular functions are approximated correctly.

NURBS discretization: numerical results

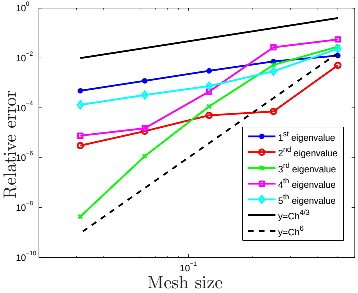
Buffa, Sangalli, V. In preparation

The domain Ω is described with a NURBS mapping.

Solution of the mixed formulation with a NURBS discretization.



NURBS map of the coarsest mesh



- The results are **spurious-free**, with optimal convergence rate.
- Also the singular functions are approximated correctly.

The spectral correctness can also be proved for (mixed) NURBS.

Conclusions

- Isogeometric Analysis has been extended to the approximation of vector fields.
  - Generalization of edge and face elements
  - Higher continuity than standard finite elements.
  - Exact geometry description.

Conclusions

- Isogeometric Analysis has been extended to the approximation of vector fields.
  - Generalization of edge and face elements
  - Higher continuity than standard finite elements.
  - Exact geometry description.
- Complete theory for Maxwell, and promising numerical results.

Conclusions

- Isogeometric Analysis has been extended to the approximation of vector fields.
  - Generalization of edge and face elements
  - Higher continuity than standard finite elements.
  - Exact geometry description.
- Complete theory for Maxwell, and promising numerical results.
- Further comparisons with FEM will be done in the future.

Conclusions

- Isogeometric Analysis has been extended to the approximation of vector fields.
  - Generalization of edge and face elements
  - Higher continuity than standard finite elements.
  - Exact geometry description.
- Complete theory for Maxwell, and promising numerical results.
- Further comparisons with FEM will be done in the future.

COMING SOON

The GeoPDEs code: a research tool for Isogeometric Analysis of PDEs.
Open OCTAVE (compatible with MATLAB) implementation of the method.
Joint work with C. de Falco and A. Reali.
http://www.imati.cnr.it/geopdes

Acknowledgments

Funded by **ERC Starting Grant n 205004:** GeoPDEs

Team

- Annalisa Buffa (IMATI-CNR)

- Lourenço Beirão da Veiga (University of Milano)
- Alessandro Reali (University of Pavia)
- Giancarlo Sangalli (University of Pavia)

- Andrea Bressan
- Durkbin Cho
- Carlo De Falco
- Mukesh Kumar
- Massimiliano Martinelli
- Judith Rivas
- Rafael Vázquez

Acknowledgments

Funded by **ERC Starting Grant n 205004:** GeoPDEs

Team

- Annalisa Buffa (IMATI-CNR)

- Lourenço Beirão da Veiga (University of Milano)
- Alessandro Reali (University of Pavia)
- Giancarlo Sangalli (University of Pavia)

- Andrea Bressan
- Durkbin Cho
- Carlo De Falco
- Mukesh Kumar
- Massimiliano Martinelli
- Judith Rivas
- Rafael Vázquez

Thanks for your attention!

http://www.imati.cnr.it/geopdes