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Non Trivial Coexistence Conditions for a Model of Language Competition Obtained by Bifurcation Theory

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Abstract We consider a modification of the model proposed by Abrams and Strogatz to describe the death of a language when it competes with a stronger one within the same community of speakers. The modification opened the possibility of coexistence of both languages under some conditions, but so far it has not been possible to write down the expression of the equilibrium points. In this paper, we nontrivially use bifurcation theory to calculate under which conditions such coexistence arises; namely, we calculate the specific ranges of the parameters that describe the modified model to have this situation, paying special attention to the cases that yield a stable cohabitation of two monolingual populations along with a bilingual one.

Keywords Bilingualism · Language competition · Non-linear dynamics

1 Introduction

The study of the interactions between human groups speaking different languages has attracted increasing attention from the scientific community in the last decade [1–3]. Espe-

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cially appealing are the cases where two languages are present on the same geographical area and a sort of competition takes place among them [4, 5]. Such tensions, in fact, may underlie several political problems in some countries. It is important, therefore, to know whether there is any mechanism that can be described in a scientific way and, more important, to know if the mechanism allows us to make estimations of future situations on the basis of such description.

Among the several approaches to the problem, a paper by Abrams and Strogatz [6] considered it from a macroscopic point of view, in the following way: two languages X, Y compete for speakers and form two monolingual populations denoted by x and y respectively, with $x + y = 1$ (the total population is normalized to 1). In their model, the shifts between groups are caused by the attractiveness of each language: this attractiveness increases with the number of speakers and with the level of the perceived status. By perceived status we mean a parameter that expresses the social and economic benefits derived from speaking that language. The probability of an individual shift from language X to Y is given by $P_{xy}(x, s)$, where $s \in [0, 1]$ is a measurement of the relative status of language X. Then their model can be written as:

$$\dot{x} = yP_{yx}(x, s) - xP_{xy}(x, s), \tag{1}$$

where $y = 1 - x$ and $P_{xy}(x, s) = P_{yx}(1 - x, 1 - s)$. Moreover, they assumed that the transition functions take the form

$$P_{yx}(x, s) = cx^a s, \qquad P_{xy}(x, s) = c(1 - x)^a(1 - s), \tag{2}$$

where c is a positive constant. These functions join the two criteria that move the decision to switch from a group to the other: the size of the group and the social status of the language.

In order to test the model they analyzed 42 different regions where two languages compete and surprisingly they found that the parameter a is quite constant

$$a = 1.31 \pm 0.25. \tag{3}$$

The system has three fixed points: $x_1 = 0$, $x_2 = 1$ and $x_3 \in [0, 1]$ where only x_1 and x_2 are stable. Then, the Abrams–Strogatz model predicts that one of the two languages goes to extinction.

The model does not take bilingualism into account (see [7] and [8] for a discussion on the role of bilingualism), which is a feature observed, for example, in some communities with more than one official language, like many regions in Spain such as Galicia (see for instance [9]), Basque Country and Catalonia, or other places in Europe [10].

A new interpretation of the Abrams–Strogatz model has been proposed by Castelló et al. [3, 11] in order to address cases of effective coexistence of groups of monolingual speakers ($s = 1/2$ or $a < 1$), although such parametric setup has not been observed in any real data set analyzed so far.

A different model has been proposed by Pinasco and Romanelli [12], for which coexistence is possible. The model is of Volterra–Lotka type:

$$\dot{X} = cXY + \alpha_x X\left(1 - \frac{X}{S_x}\right),$$
$$\dot{Y} = -cXY + \alpha_y Y\left(1 - \frac{Y}{S_y}\right),$$

where c is the rate of conversion from language y to x, the parameters α_x, α_y represent natality and mortality rates of each population and S_x, S_y are the carrying capacities in absence of competition. In particular they proved the existence of a stable fixed point (under the condition $S_x < \frac{\alpha_y}{c}$) in the first quadrant with no zero coordinates. But, again for these kind of models, bilingual individuals are basically ignored, i.e., bilinguals are not considered as an specific group in the calculations.

In order to incorporate them, for example, Baggs and Freedman have used a predator–prey model, first describing the interaction of a bilingual with a monolingual population [13] (see also [14]) and, later, generalizing the model considering two monolingual and one bilingual populations and studied the persistence of the system [15].

A different approach was used by Mira and Paredes [16], who modified the Abrams–Strogatz model in order to describe situations in which bilingualism is stable. They introduced a bilingual population b such that $x + y + b = 1$ and a new parameter $k \in [0, 1]$ that represents the interlinguistic similarity; that is, $k = 0$ means that there is no similarity, whereas $k = 1$ implies $X = Y$. The system takes the form:

$$\begin{aligned} \dot{x} &= y P_{YX} + b P_{BX} - x(P_{XY} + P_{XB}), \\ \dot{y} &= x P_{XY} + b P_{BY} - y(P_{YX} + P_{YB}), \\ \dot{b} &= x P_{XB} + y P_{YB} - b(P_{BY} + P_{BX}), \end{aligned} \tag{4}$$

with the transition functions:

$$\begin{aligned} P_{XB} &= c \cdot k(1-s)(1-x)^a, \\ P_{YB} &= c \cdot ks(1-y)^a, \\ P_{BX} &= P_{YX} = c \cdot (1-k)s(1-y)^a, \\ P_{BY} &= P_{XY} = c \cdot (1-k)(1-s)(1-x)^a. \end{aligned}$$

The reason for introducing the parameter k is motivated by the conjecture that a great similarity (high value of the parameter k) between the languages in competition facilitates bilingualism. Mira and Paredes tested their model with real data from Galicia, a region with a large bilingual population, and, from fits to historical data of percentages of speakers, they fitted successfully such historical trend and found experimentally the value $k = 0.80$ for the pair Galician–Castillian Spanish, a similarity in accordance with estimates obtained from other linguistic perspectives [17]. The model (4) was analyzed later in [18], presenting many numerical simulations in order to establish the possibility of coexistence of the two languages.

They found that the asymptotic dynamics might lead to the survival of both languages (both in monolingual groups of speakers as well as within a community of bilinguals) or to the extinction of the weakest tongue depending on the different parameters. This was followed by a study of the coexistence from an analytical point of view [19]. It was shown that coexistence is possible when three fixed points appear inside the region

$$A := \left\{(x, y) \in \mathbb{R}^2 : \ x, y \geq 0, \ x + y \leq 1\right\}. \tag{5}$$

It was also shown that it is possible to almost completely assay the number and nature of the equilibrium points of the model, which depends on its parameters, as well as to build a phase space based on them. This information is crucial in order to study how the languages evolve with time. The rigorous considerations also suggested ways to further improve the model and facilitate the comparison of its consequences with those from other approaches or with real data. In [20] the behavior of the system was analyzed for $a = 1$ and the authors derived some necessary conditions to solve the optimal control problem and presented some numerical simulations.

Whereas in [19] the authors obtain a theoretical result about coexistence (coexistence is possible), here we are interested in determining how coexistence solutions depend on the parameters $k \in [0, 1]$ and $s \in [0, 1]$ in order to analyze real data and predict the behavior of real situations of competition. Since it is not possible to write down the expression of the fixed points, we use bifurcation theory to find the bifurcating curves $s(k)$ for any values of $a \in (1, 2)$. By estimating the values of the similarity $\bar{k}$ between two languages in competition, we are able to obtain explicitly the optimal interval $(s_1(\bar{k}), s_2(\bar{k}))$ in which the system admits coexistence and, as a consequence, this result could stimulate proper policies for saving languages under risk. Throughout the paper we consider the parameter $a \in (1, 2)$, which is the interesting range for real applications (see (3)). The figures presented are plotted for $a = 1.31$, the average exponent obtained from real data. For the sake of simplicity, we put $c = 1$ without losing generality.

We will provide an exhaustive analysis of the system (4) in the following sections. In details, the rest of the paper is organized as follows: in Sect. 2 we present some preliminary considerations about fixed points and nullclines, while in Sect. 3 we analyze a particular case of coexistence, in which the system undergoes a pitchfork bifurcation. In Sect. 4 we study the general problem of coexistence depending on parameters (k, s). In Sect. 5 we discuss the results of the present paper for the case of competition between Galician and Castillian Spanish, while in the last section we give some remarks and suggest some further investigation.

2 Preliminary Considerations

In this section we work on some basic features of the model. The third differential equation that tracks the evolution of the proportion b of bilinguals in the equations (4) will not be needed in the following work, thanks to the hypothesis that the population size is constant. For simplicity we normalize the population $P(t) = x + y + b = 1$, then the system takes the form:

$$\dot{x} = P_{BX} + y(P_{YX} - P_{BX}) - x(P_{XY} + P_{XB} + P_{BX}),$$
$$\dot{y} = P_{BY} + x(P_{XY} - P_{BY}) - y(P_{YX} + P_{YB} + P_{BY}).$$

By writing in detail the transition functions we obtain:

$$\begin{aligned} \dot{x} &= c(1-x)\big\{(1-k)s(1-y)^a - (1-s)x(1-x)^{a-1}\big\}, \\ \dot{y} &= c(1-y)\big\{(1-k)(1-s)(1-x)^a - sy(1-y)^{a-1}\big\}. \end{aligned} \tag{6}$$

Since we are interested in the coexistence of the three populations x, y and b, we limit our analysis to the set A (see (5)) that is positively invariant (see [19] for a detailed proof).

For any values of the parameters $k, s \in [0, 1]$ the system admits three fixed points:

$$P_1 = (1, 1), \qquad P_2 = (1, 0), \qquad P_3 = (0, 1). \tag{7}$$

As P_1 is outside A and the points P_2, P_3 are in ∂A, this means that there always exist positive solutions (starting sufficiently close to P_2 or P_3 inside A) which leads to the extinction of two of the three populations (one bilingual and one monolingual). If $k = 1$ and $s \neq 0, 1$ we have the further fixed point $P_0 = (0, 0)$ that is stable and there exists an heteroclinic connection between the 4 fixed points, that is, $y = 1$ is a stable curve (see [21]) for P_3 and unstable for P_1; $x = 1$ is a stable curve for P_2 and unstable for P_1, the x and y axes are stable curves for P_0 and unstable for P_2 and P_3 respectively. If $k = 1$ and $s = 0$, we have two vertical segments of fixed points: one segment of stable fixed points connecting P_0 and P_3 and another one of unstable fixed points connecting P_1 and P_2. The solutions consist of horizontal lines which converge to a stable fixed point in the future and to an unstable fixed point in the past. If $k = 1$ and $s = 1$ we have a similar situation: two horizontal segments of fixed points, the one connecting P_0 and P_2 is stable, whereas the other is unstable. If $k \neq 0, 1$ and $s = 0$ or $s = 1$ the region A is no longer positive invariant and it is possible to obtain the solution curves of the system. If $s = 0$ the system becomes:

$$\begin{cases} \dot{x} = -x(1-x)^a, \\ \dot{y} = (1-k)(1-y)(1-x)^a, \end{cases} \tag{8}$$

then, by considering the quotient of the two equations we deduce

$$\frac{\dot{y}}{\dot{x}} = \frac{dy}{dx} = \frac{(1-y)(1-k)}{-x},$$

and integrating we obtain the explicit solution:

$$y(x) = 1 - (1-y_0)\left(\frac{x}{x_0}\right)^{1-k}.$$

In the case in which $s = 1$ we have:

$$\begin{cases} \dot{y} = (1-k)(1-x)(1-y)^a, \\ \dot{y} = -y(1-y)^a, \end{cases} \tag{9}$$

then, by considering the quotient of the two equations we obtain

$$\frac{\dot{y}}{\dot{x}} = \frac{dy}{dx} = \frac{-y}{(1-x)(1-k)},$$

and integrating:

$$y(x) = y_0\left(\frac{1-x}{1-x_0}\right)^{\frac{1}{1-k}}.$$

As a consequence of the previous considerations we will look for coexistence solutions when:

$$k \neq 1, \qquad s \notin \{0, 1\}. \tag{10}$$

We call the functional Jacobian of the vector field, $\mathcal{A} = ((a_{i,j}))$. In detail we have:

$$a_{11} = -(1-k)s(1-y)^a - (1-s)(1-x)^a + a(1-s)x(1-x)^{a-1},$$
$$a_{12} = -as(1-k)(1-x)(1-y)^{a-1},$$

$$a_{21} = -a(1-k)(1-s)(1-y)(1-x)^{a-1},$$
$$a_{22} = -(1-k)(1-s)(1-x)^{a} - s(1-y)^{a} + asy(1-y)^{a-1}.$$

The matrices of the linearized system at point P_2 and P_3 are respectively:

$$\mathcal{A}(P_2) = \begin{pmatrix} -(1-k)s & 0 \\ 0 & -s \end{pmatrix}, \qquad \mathcal{A}(P_3) = \begin{pmatrix} -(1-s) & 0 \\ 0 & -(1-k)(1-s) \end{pmatrix},$$

whereas at point P_1, $\mathcal{A}(P_1)$ is the 0 matrix. If (10) holds, then the eigenvalues of the matrices $\mathcal{A}(P_2)$ and $\mathcal{A}(P_3)$ are all negative. As a consequence, the points P_2 and P_3 are asymptotically stable; this means that, if the solution starts in a sufficient small neighborhood of P_i (for $i = 2, 3$), then they converge to it, that is, we have the persistence of only one language (X or Y respectively).

The line $y = 1$ is a stable curve for the point P_3 and an unstable curve for P_1, whereas the line $x = 1$ is a stable curve for P_2 and an unstable one for P_1, then the point P_1 is unstable. In cite [19] it was proved that the system admits 1, 2 or 3 more fixed point in A and that coexistence is possible if and only if we have three fixed point inside A.

In the present work we provide necessary and sufficient conditions to have the existence of three fixed points inside A. In order to do that we study the problem of bifurcation, that is how the systems pass from 2 fixed point to three fixed points inside A.

The fixed points inside A are generated by the intersection of the parabola-like nullclines (see Fig. 1 below):

$$\begin{aligned} n_x: \quad & y = 1 - \left[\frac{(1-s)}{(1-k)s} x(1-x)^{a-1}\right]^{1/a}, \\ n_y: \quad & x = 1 - \left[\frac{s}{(1-k)(1-s)} y(1-y)^{a-1}\right]^{1/a}. \end{aligned} \tag{11}$$

The first one passes through the fixed points P_1 and P_3 and it is contained between the lines $x = 0$ and $x = 1$, whereas the second curve passes through the fixed points P_1 and P_2 and it is contained between the lines $y = 0$ and $y = 1$. It is useful to consider the vertexes (minima) $V_1 = (x_{v1}, y_{v1})$ and $V_2 = (x_{v2}, y_{v2})$ of the nullclines:

$$x_{v1} = \frac{1}{a}, \qquad y_{v1} = 1 - \left\{\frac{1-s}{(1-k)s} \frac{(a-1)^{a-1}}{(a)^{a}}\right\}^{1/a}, \tag{12}$$

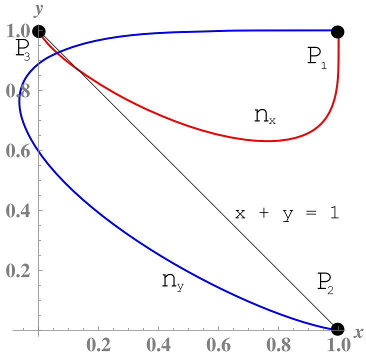

Fig. 1 The parabolic nullclines with $k = \frac{1}{10}$ and $s = \frac{2}{3}$, $a = 1.31$. In this case there exists only one fixed point inside the region A

$$y_{v2} = \frac{1}{a}, \qquad x_{v2} = 1 - \left\{ \frac{s}{(1-k)(1-s)} \frac{(a-1)^{a-1}}{(a)^a} \right\}^{1/a}. \tag{13}$$

Then, for all value of k and s, the vertexes are respectively on the lines $x = \frac{1}{a}$ and $y = \frac{1}{a}$ and, as a consequence, the first curve is decreasing for $x \in [0, \frac{1}{a}]$ and increasing for $x \in [\frac{1}{a}, 1]$, whereas the second is increasing for $y \in [0, \frac{1}{a}]$ and decreasing for $y \in [\frac{1}{a}, 1]$.

We observe that, as $k \to 1$, the vertex of the first parabola moves downwards, whereas the other vertex moves to the left of the y axes. This fact is due to the dependence on k, that is of the form $(1-k)^{-\frac{1}{a}}$. This means that the number of fixed points increases as k increases (see [19] for a complete discussion).

We end this section by describing the case $k = 0$. For this purpose we rewrite the intersection between the parabolic nullclines in the following way:

$$(1-k)^2(1-x)(1-y) = xy, \tag{14}$$

$$(1-s)^2 x(1-x)^{2a-1} = s^2 y(1-y)^{2a-1}. \tag{15}$$

Then, if $k = 0$, using (14) we have that the intersection is on the line $x + y = 1$ with coordinates:

$$P = \left(1 - \frac{1}{1 + (\frac{1-s}{s})^{\frac{1}{a-1}}}, \frac{1}{1 + (\frac{1-s}{s})^{\frac{1}{a-1}}}\right), \quad s \neq 0.$$

We will see later that these points are unstable, in any case, since they are on $x + y = 1$, we have the extinction of the bilingual group. In the following sections we will limit our analysis to the set A.

3 Coexistence and Pitchfork Bifurcation for $s = \frac{1}{2}$

Since the system is symmetric with respect to the transformation $(s, x, y) \to (1-s, y, x)$, when $s = 1 - s = \frac{1}{2}$ it is possible to interchange x and y without changing the equations of the system. The parabolic nullclines are symmetric with respect to the line $x = y$ and, as a consequence, there exists a fixed point, that we call P_4, that lies on the line $x = y$:

$$P_4 = \left(\frac{1-k}{2-k}, \frac{1-k}{2-k}\right). \tag{16}$$

It is easy to see that the point P_4 is always inside A for any $k \neq \{0, 1\}$. In fact we have that

$$0 < \frac{1-k}{2-k} < \frac{1}{2}. \tag{17}$$

In order to study the stability of the fixed point, we consider the linearized system. The eigenvalues and eigenvectors of the matrix of the linearized systems at P_4 are:

$$\lambda_1 = -\frac{1}{2}\left(\frac{1}{2-k}\right)^{a-1}, \qquad v_1 = (1, 1), \tag{18}$$

$$\lambda_2 = -\frac{1}{2}\left(\frac{1}{2-k}\right)^{a}\left[(2a-1)k + 2 - 2a\right], \qquad v_2 = (1, -1). \tag{19}$$

The eigenvalue λ_1 is negative for any value of k whereas $\lambda_2 < 0$ if

$$k > \frac{2a-2}{2a-1}. \tag{20}$$

As a summary of the analysis of the present section, we can state the following:

Theorem 1 *If $s = 1/2$ and k satisfies (20) the system (6) admits coexistence of the three populations.*

We recall that $s = 1/2$ means that the two languages X and Y have equivalent status. We expect that the coexistence solutions are produced by a subcritical Pitchfork bifurcation, with the bifurcation value given by:

$$k^* = \frac{2a-2}{2a-1}.$$

For simplicity we rewrite the system in the following way;

$$\begin{cases} \dot{x} = f(x, y) := \dfrac{1}{2}\{(1-k)(1-x)(1-y)^a - x(1-x)^a\}, \\ \dot{y} = g(x, y) := \dfrac{1}{2}\{(1-k)(1-y)(1-x)^a - y(1-y)^a\}. \end{cases} \tag{21}$$

In order to prove that there is Pitchfork bifurcation we consider the following theorem by Sotomayor (see [22]):

Theorem 2 *Suppose that $F(X_0, \mu_0) = 0$ and that the $n \times n$ matrix $\mathcal{A} = DF(X_0, \mu_0)$ has a simple eigenvalue $\lambda = 0$ with eigenvector v and that $\mathcal{A}^T$ has an eigenvector w corresponding to the eigenvalue $\lambda = 0$. Suppose that the matrix $\mathcal{A}$ has k eigenvalues with negative real parts and $n - k - 1$ eigenvalues with positive real part and that the following conditions are satisfied:*

$$w^T F_\mu(X_0, \mu_0) = 0, \tag{22}$$

$$w^T \left[DF_\mu(X_0, \mu_0)v\right] \neq 0, \tag{23}$$

$$w^T \left[D^2 F_\mu(X_0, \mu_0)v, v\right] = 0, \tag{24}$$

$$w^T \left[D^3 F_\mu(X_0, \mu_0)v, v, v\right] \neq 0. \tag{25}$$

Then the system $\dot{X} = F(X, \mu)$ experiences a Pitchfork bifurcation at the equilibrium point X_0 as the parameter μ varies through the bifurcation value $\mu = \mu_0$.

In this case of system (21) we have that $v = (v_1, v_2) = w = (1, -1)$, in the following lines we are verifying hypotheses (22)–(25). We have that

$$f_k(x, y) := -\frac{1}{2}(1-x)(1-y)^a,$$

$$g_k(x, y) := -\frac{1}{2}(1-y)(1-x)^a,$$

then we obtain (22):

$$(1,-1)\cdot\big(f_k\big(P_4,k^*\big),g_k\big(P_4,k^*\big)\big)=0.$$

Now we compute:

$$DF_k=\begin{pmatrix}\frac{1}{2}(1-y)^a & \frac{1}{2}a(1-x)(1-y)^{a-1}\\ \frac{1}{2}a(1-y)(1-x)^{a-1} & \frac{1}{2}(1-x)^a\end{pmatrix},$$

then

$$\begin{aligned}w^T\big[DF_k(x,y)v\big]&=\frac{1}{2}\big[(1-x)^a+(1-y)^a\big]\\&\quad-\frac{1}{2}a\big[(1-x)(1-y)^{a-1}+(1-y)(1-x)^{a-1}\big],\end{aligned}$$

and computed at (P_4,k^*) it gives (23):

$$-\frac{a-1}{(2-k^*)^a}\neq 0.$$

Now we pass to check (24):

$$\begin{aligned}D^2F_k\big(P_4,k^*\big)v,v&=\begin{pmatrix}\frac{\partial^2}{\partial x^2}f_k(P_4,k^*)v_1^2+2\frac{\partial^2}{\partial x\partial y}f_k(P_4,k^*)v_1v_2+\frac{\partial^2}{\partial y^2}f_k(P_4,k^*)v_2^2\\ \frac{\partial^2}{\partial x^2}g_k(P_4,k^*)v_1^2+2\frac{\partial^2}{\partial x\partial y}g_k(P_4,k^*)v_1v_2+\frac{\partial^2}{\partial y^2}g_k(P_4,k^*)v_2^2\end{pmatrix}\\&=\begin{pmatrix}\frac{a}{(2-k^*)^{a-1}}-\frac{1}{2}\frac{a(a-1)}{(2-k^*)^{a-1}}\\ \frac{a}{(2-k^*)^{a-1}}-\frac{1}{2}\frac{a(a-1)}{(2-k^*)^{a-1}}\end{pmatrix},\end{aligned}$$

then we obtain (24):

$$(1,-1)\cdot\big[D^2F_k\big(P_4,k^*\big)v,v\big]=0.$$

We conclude by checking (25). We have:

$$\begin{aligned}&D^3F_k\big(P_4,k^*\big)v,v,v\\&=\begin{pmatrix}\frac{\partial^3}{\partial x^3}f_k(P_4,k^*)v_1^3+\frac{\partial^3}{\partial x^2\partial y}f_k(P_4,k^*)v_1^2v_2+\frac{\partial^3}{\partial x\partial y^2}f_k(P_4,k^*)v_1v_2^2+\frac{\partial^3}{\partial y^3}f_k(P_4,k^*)v_2^3\\ \frac{\partial^3}{\partial x^3}g_k(P_4,k^*)v_1^3+\frac{\partial^3}{\partial x^2\partial y}g_k(P_4,k^*)v_1^2v_2+\frac{\partial^3}{\partial x\partial y^2}g_k(P_4,k^*)v_1v_2^2+\frac{\partial^3}{\partial y^3}g_k(P,k^*)v_2^3\end{pmatrix}\\&=\begin{pmatrix}\frac{1}{2}\frac{a(a-1)}{(2-k^*)^{a-2}}-\frac{1}{2}\frac{a(a-1)(a-2)}{(2-k^*)^{a-2}}\\ \frac{1}{2}\frac{a(a-1)(a-2)}{(2-k^*)^{a-2}}-\frac{1}{2}\frac{a(a-1)}{(2-k^*)^{a-2}}\end{pmatrix},\end{aligned}$$

from which we obtain conditions (25):

$$w^T\big[D^3F_k\big(P_4,k^*\big)v,v,v\big]=\frac{a(a-1)(3-a)}{(2-k^*)^{a-2}}\neq 0.$$

Then we can conclude (see Fig. 2):

Theorem 3 *The system* (21) *experiences a Pitchfork Bifurcation for the fixed point P_4 as the parameter k pass trough the bifurcation value k^*.*

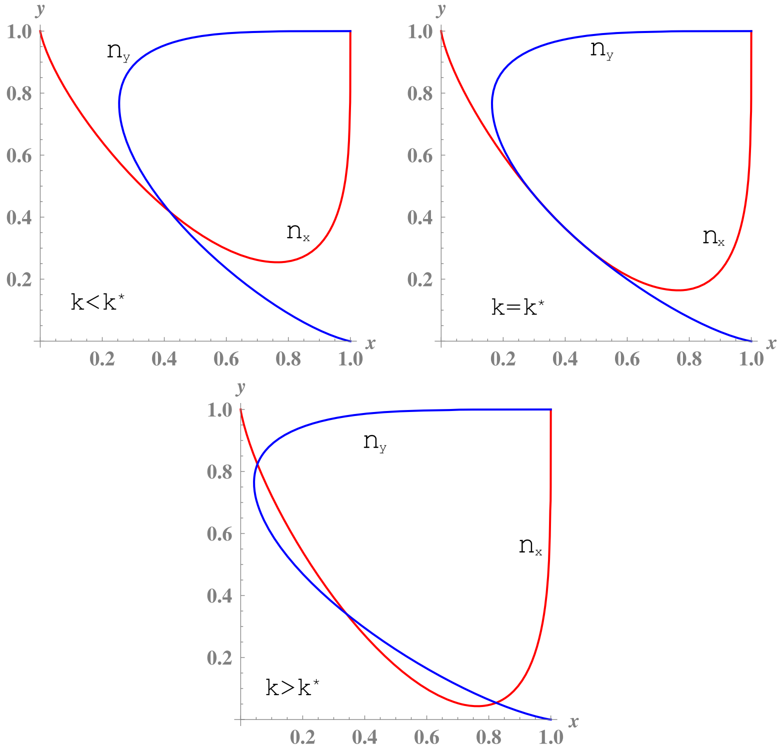

Fig. 2 The pitchfork bifurcation scenario: as the parameter k passes the critical value k^*, for which we have tangency of the parabolic nullclines, the unique unstable fixed point becomes stable and two unstable fixed points appear

4 Coexistence for $s \in [0, 1]$

In the previous section we have shown that if $s = \frac{1}{2}$ the nullclines are symmetric with respect to $x = y$ and there always exists a fixed point on the line $x = y$ which undergoes a Pitchfork bifurcation at $k = k^*$. For $(k, s) = (k^*, \frac{1}{2})$ the parabolic nullclines are tangent. Two new fixed points appear inside A when k crosses the critical value k^* and the fixed point on the line $x = y$ gains stability. In this section we study the problem of the bifurcation of the fixed points in the general case, in which $s \in [0, 1]$. From a geometrical point of view, the fixed points bifurcate when the nullclines are tangent and this happens when the decreasing part of the first parabolic nullcline is tangent to the increasing part of the second nullcline. This is the only interesting case, in which the tangent vectors of the nullclines can be parallel. The other case, that is, the increasing part of the first nullcline with the decreasing part of the second nullcline, is geometrically impossible due to the convexity of the nullclines and of their intersection at P_1. When $s \neq \frac{1}{2}$, the tangency occurs before the vertex of the first nullcline and after the vertex of the second nullcline; as a consequence, from monotonicity arguments there must be another intersection between the nullclines. Then, when $s \neq \frac{1}{2}$, there is one fixed point after a critical value of the parameter k and another fixed point appears, that is, the point where the nullclines are tangent bifurcates to two more fixed points when the critical value is crossed. This scenario is mathematically defined as a saddle–node bifurcation as observed in [19] (see Fig. 3). Then, for values of k greater than the critical

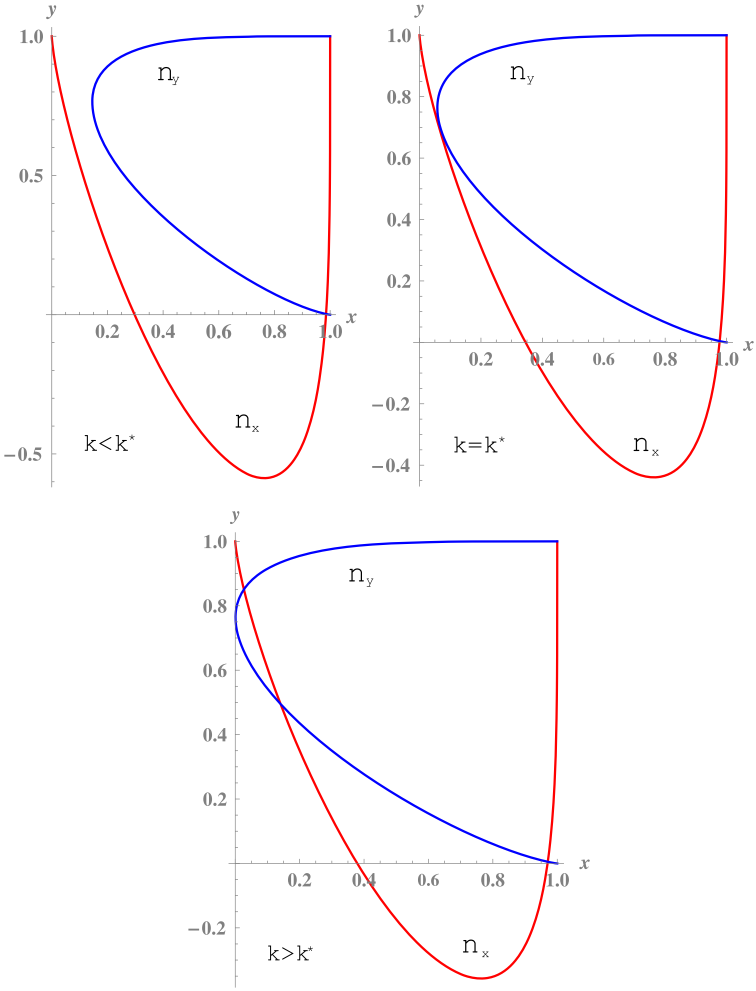

Fig. 3 The saddle–node bifurcation: a fixed point, obtained by the tangency of nullclines, appears for the critical value of the parameter and then it bifurcates to two fixed points, one unstable and the other stable

values $k^*(s)$, we obtain three fixed points inside A, one of them stable and, as a consequence, we arrive to the coexistence of solutions. In order to completely understand the problem of coexistence we will find the expression of the critical values $k^*(s)$, which represents the boundary of the region of coexistence in the set $(k, s) \in [0, 1] \times [0, 1]$. In order to study the region of coexistence we will consider two conditions of bifurcation

1. Analytical condition: one of the eigenvalues is zero.
2. Geometrical condition: at the fixed point the nullclines are tangent.

We start analyzing the first condition: for this purpose, we rewrite the Jacobian matrix at a generic fixed point using (11):

$$\mathcal{A} = s(1-y)^{a-1}B,$$

where the matrix B has the following entries

$$
\begin{aligned}
b_{11} &:= (a-1)(1-k)(1-y) - \frac{y}{1-k},\\
b_{12} &:= -a(1-k)(1-x),\\
b_{21} &:= -a\frac{y(1-y)}{1-x},\\
b_{22} &:= ay-1.
\end{aligned}
$$

Then we have that the trace of A satisfies:

$$Tr(A) = s(1-y)^{a-1}Tr(B),$$

where

$$Tr(B) = y\left(1 - \frac{1}{1-k} + k(a-1)\right) + (a-2) - k(a-1).$$

If $y \neq 1$, the sign of the trace of A is the same as that of the trace of B, then we observe that

$$(a-2) - k(a-1) < 0, \quad \forall k \in [0,1],$$

since $a \in (1,2)$. Moreover

$$1 - \frac{1}{1-k} + k(a-1) = -\frac{k[a+k(a-1)]}{1-k} < 0, \quad \forall k \in (0,1].$$

Then

$$Tr(B) < 0, \quad \forall k \in [0,1], \quad \forall y \in [0,1],$$

and, as a consequence

$$Tr(\mathcal{A}) < 0, \quad \forall k \in [0,1], \quad \forall y \in [0,1), \quad s \neq 0.$$

This means that at least one of the eigenvalues is always negative and the problem of coexistence depends on the sign of the other eigenvalue. Then we will have bifurcation whenever the second eigenvalue passes trough zero, that is, when:

$$det(\mathcal{A}) = 0 \quad \Longleftrightarrow \quad det(B) = 0 \quad \text{or} \quad y = 1.$$

Of course we can avoid the case $y = 1$, which corresponds to the fixed points P_1, P_3 analyzed in Sect. 2. Then, we consider

$$det(B) = (1-k)(1-y)(1-a-ay) - \frac{y(ay-1)}{1-k} = 0,$$

from which we obtain

$$y_{1,2} = \frac{1}{2a}\left(1 \mp \sqrt{(2a-1)^2 - \frac{4a(a-1)}{k(2-k)}}\right).$$

In order to find the x-components of the bifurcating fixed points we use the second condition, the tangency of the nullclines. In order to obtain tangency we need that the tangent unit vectors of the nullclines are the same at the fixed points. We implicitly derive the expression of the nullclines obtaining

$$(1-k)sa(1-y)^{a-1}y' + (1-s)\left\{(1-x)^{a-1} - x(a-1)(1-x)^{a-2}\right\} = 0,$$

and

$$(1-k)(1-s)a(1-x)^{a-1}(-1) - sy'(1-y)^{a-1} - sy(a-1)(1-y)^{a-2}\left(-y'\right) = 0.$$

Then, using the first equation we derive the expression of y' and substituting it in the second one we obtain the following curve:

$$(ax-1)(ay-1) = a^2(1-k)^2(1-x)(1-y). \tag{26}$$

Of course this is not a sufficient condition of tangency, in fact we need also that the nullclines intersect. Using (14) and (26) we conclude that the bifurcating fixed points lie on the line

$$x + y = \frac{1}{a}. \tag{27}$$

Finally, using the previous condition we are able to write the x-components of the bifurcating fixed points:

$$x_{1,2} = \frac{1}{a} - y_{1,2} = \frac{1}{2a}\left(1 \pm \sqrt{(2a-1)^2 - \frac{4a(a-1)}{k(2-k)}}\right).$$

We observe that $x_1 = y_2$ and $x_2 = y_1$ and, since the system is symmetric under the transformation $(s, x, y) \to (1-s, y, x)$, we have that, if (x_1, y_1) corresponds to the values s of the status parameter, then (x_2, y_2) corresponds to the value $1-s$. It is easy to check that $x_i, y_i > 0$ for all $k \in (0, 1)$ and (x_i, y_i) is inside A since $x_i + y_i = \frac{1}{a}$ and $a \in (1, 2)$.

We observe that by this expression we recover the case studied in the previous section where $x = y = \frac{1}{2a}$ (corresponding to $s = \frac{1}{2}$ and $k = \frac{2a-2}{2a-1}$); moreover, the fixed points (x_1, y_1) and (x_2, y_2) exist when

$$(2a-1)^2 - \frac{4a(a-1)}{k(2-k)} \geq 0 \quad \iff \quad k \in \left[\frac{2a-2}{2a-1}, \frac{2a}{2a-1}\right] \cap [0, 1] = \left[\frac{2a-2}{2a-1}, 1\right].$$

We note that for the case $s = \frac{1}{2}$ we have the lowest critical value of k, moreover, to any critical value k^* correspond the values s^* and $1-s^*$ of the other parameter.

In order to find the values of s corresponding to each fixed point we consider the expression of one of the nullclines, from which we find:

$$s = f_i(k) := \frac{1}{1 + \frac{(1-k)(1-y_i)^a}{x_i(1-x_i)^{a-1}}}, \quad i = 1, 2. \tag{28}$$

In Fig. 4 we represent the functions f_1 and f_2 for $a = 1.31$. The function f_1 corresponds to values of $s \in [\frac{1}{2}, 1]$ and the function f_2 corresponds to the values of $s \in [0, \frac{1}{2}]$. Then, to any $k \geq \frac{2a-2}{2a-1}$, we have two fixed points (x_1, y_1) and (x_2, y_2) corresponding to values of s given by $f_1(k)$ and $f_2(k)$ respectively (if $s = \frac{1}{2}$ we have that $f_1(k) = f_2(k)$).

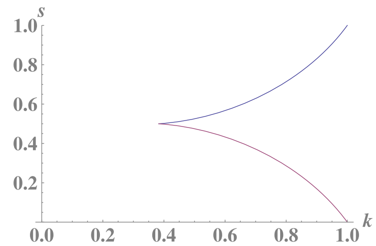

Fig. 4 The functions f_1(*upper branch*) and f_2 (*lower branch*) which represent the critical values of the parameters $(k, s) \in [0, 1] \times [0, 1]$ for pitchfork bifurcation (if $s = \frac{1}{2}$) and for saddle node bifurcation (if $s \neq \frac{1}{2}$). We represent the case $a = 1.31$

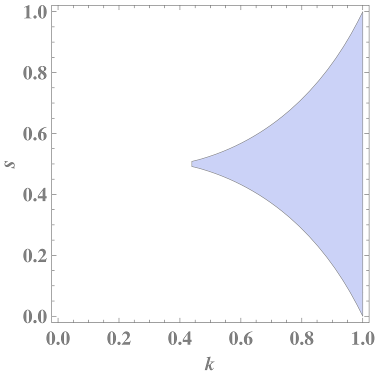

Fig. 5 The coexistence region for the parameter k (*horizontal axis*) and s (*vertical axis*). From the symmetry of the problem the region results symmetric with respect to the horizontal line $s = \frac{1}{2}$. We represent the interesting case $a = 1.31$

Using the above function we have that, for any s, we obtain the critical value of k for which we have bifurcation, that is, $f_i^{-1}(s)$, with $i = 1$ if $s \in [\frac{1}{2}, 1]$ and $i = 2$ if $s \in [0, \frac{1}{2}]$. Then, the region of coexistence is given by:

$$\left\{(k, s) \in [0, 1] \times [0, 1] : \ f^{-1}(s) < k < 1\right\},$$

where we have set

$$f^{-1}(s) := \begin{cases} f_1^{-1}, & \text{if } s \in [\frac{1}{2}, 1], \\ f_2^{-1}, & \text{if } s \in [0, \frac{1}{2}]. \end{cases}$$

In Fig. 5 we represent the coexistence region for $a = 1.31$.

5 A Discussion of the Case of Galician–Spanish

In the paper [16] the model (6) has been tested for the case of Galician–Spanish. In particular, the parameters of the system have been computed experimentally (see [18]):

$$s = 0.26, \qquad k = 0.80. \tag{29}$$

In Fig. 6 the vector field and the nullclines are represented. There exists an unstable fixed point inside A very close to the stable fixed point $P_2 = (1, 0)$. There are no solutions of coexistence, the basin of attraction of the point P_3 is larger than that of P_2 and, since the initial conditions (obtained from [30]):

$$x = 0.89 \text{ (Galician)}, \qquad y = 0.03 \text{ (Spanish)}, \qquad b = 0.08 \text{ (Bilingual)}$$

fall in the basin of P_3, then the solutions are asymptotic to P_3.

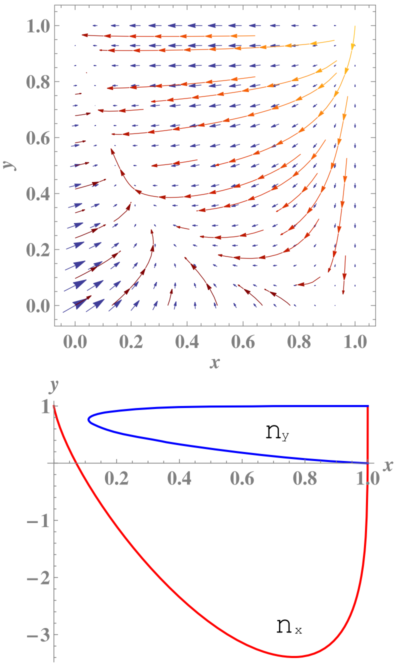

Fig. 6 The vector field in the xy plane and the nullclines of the system (6) for the choice of parameters as in (29). The basin of attraction of the fixed point $P_3 = (0, 1)$ is larger than that of $P_2 = (1, 0)$ and attracts almost all the orbits starting inside A

Using the results of the previous section we obtain that $k^* = 0.80$ is a critical value for

$$s^* \approx 0.29.$$

As the experimentally found parameter $s = 0.26$ is very close to the critical value, any strategy (of political, cultural or social type) to increase a little bit the status of Galician would have the effect of placing the system in a region of coexistence of both languages. Certainly, beyond considerations on the accuracy of the parameters, it seems clear that the system Galician–Spanish is close to a coexistence scenario.

6 Conclusion

In this paper we completely solve the problem of coexistence of the three populations for the model (6) with $a \in (1, 2)$. Even if the case studied is in the ranges observed for real cases, it is also interesting to consider the system for the case $a < 1$ (see [23]), in which the vector field is not smooth. That case require different techniques as suggested in [19].

It would be also interesting to consider the problem in the context of evolution (see [24, 25]), in particular, to study how k and s vary in time. It is worth noting that, if we are close to the bifurcation curve, small variations of the parameters would bring drastic changes in the asymptotic dynamics. The variation of the parameter k, that represents the language similarity, is related to the problem of language complexification (see for example [26]). The evolution of the parameter k would not be trivial since the languages are subjected to opposite tendencies like diversification or convergence (see [27]). It is also interesting to consider the dynamics of the status of the languages (see for instance [28] or [29]) represented in the model by the parameter s.

In conclusion, the results of this paper do not only give necessary and sufficient conditions, in the framework of this model, for the coexistence of different languages, but also provide a simple way (see (28)) to compute in detail the critical value for the parameter s to obtain coexistence in a system of two languages in competition. If the similarity between the languages is sufficiently large, that is

$$k \geq k^*(a) = \frac{2a-2}{2a-1}$$

then there exist coexistence solutions for $s \in (s^*(k), 1 - s^*(k))$ where s^* is given by (28). We observe that k^* is an increasing function of the parameter a with range $[0, \frac{2}{3}]$ when $a \in [1, 2]$.

For the value $a = 1.31$ obtained by Abrams and Strogatz, that is, $a = 1.31$, the minimum level of required similarity is $k^* \approx 0.38$ that is a relatively low value if we think of some concrete cases. Then, in some cases, coexistence between two competing languages will be impossible, whereas in other cases it will be possible to calculate exactly how far the parameter s is from the coexistence case.

As a consequence, the present study shows that it is possible to save languages under risk and, therefore, it could be a useful guide to political and cultural institutions to undertake actions to preserve a language under risk.

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