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Archimedes meets Einstein: a millennial geometric bridge

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Abstract

This contribution explores some analogies between special relativity and geometrical tools developed by the ancient Greeks. The kinematics of one-dimensional elastic collisions is solved with simple ruler-and-compass constructions on conic sections. Then, a thought-provoking relation involving Lorentz transformations, Archimedes' law of the lever and Einstein's formula for the relativistic mass is put forward. The familiarity with classical geometry is useful in developing intuitions on deep concepts of modern physics and can be profitable for high school or basic undergraduate teaching. Moreover, it is fascinating to establish a bridge connecting beautiful ideas separated by two millennia.

Keywords: special relativity, education in Physics, geometrical methods

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(Some figures may appear in colour only in the online journal)

1. Introduction

Special relativity constitutes one of the summits of modern science. Still, most of its conceptual novelties remain far from being familiar to the layperson more than a century after its inception by Albert Einstein. More than two millennia earlier, the ancient Greeks, including Euclid and Archimedes among other salient figures, developed important geometrical tools as well as a handful of notions that lie at the basis of classical physics. The layperson is by far more acquainted with these old concepts. Thus, it is appealing to attempt to establish some connection between these ideas, both with an educational goal, as well as to find a bridge linking two periods of the human culture that are arguably the most influential in building our current ideas about the physical world.

As early as in 1907, Minkowski famously introduced the notion of spacetime [1], a stage where all physical phenomena take place, replacing the Euclidean space and the universal clock customarily used since Galileo. The concept became central to the subsequent development of physics, quickly going from a mathematical tool to a physical reality. Einstein realised that energy and matter source the curvature of the spacetime fabric and move under its spell. Most strikingly, its minuscule vibrations have been recently detected in awe [2].

Minkowski soon realised that the spacetime diagram is a useful framework that leads to a visual display of events and their relationships (simultaneity, causality, time dilation) [3]. This was further developed by many people, starting from Max Born [4] and Paul Gruner [5]. Much more recently, Takeuchi [6], Mermin [7, 8] and Salgado [9], among others, have followed Minkowski's footsteps, and graphical representations of special relativity keep being an object of current interest. They are profitable with didactical purposes at the high school or undergraduate level, for science popularisation as well as for conveniently visualising otherwise abstract expressions.

Along this line, the aim of this article is to stand on earlier contributions and supplement them with some geometrical methods developed by the ancient Greeks. Concretely, the discussion makes use of properties of conic sections, ruler-and-compass constructions and Archimedes' law of the lever. It will be shown that they can be converted into useful tools for the understanding of relativistic concepts. Since didactics based on geometry is frequently used as a supplement of algebraic developments, the approach might be interesting for introductory courses on special relativity.

The particular questions of special relativity that are addressed below are the kinematics of elastic collisions (section 2) and the increase of a body's inertia due to its state of motion (section 3). Some comments regarding the educational potential of the construction are presented in section 4.

In the following, the symbol m represents the rest mass and natural units $c = 1$ are used.

2. Geometrical methods for elastic collisions

In this section, the kinematics of one-dimensional elastic collisions is considered by using geometrical methods in the momentum-energy plane [10]. Both the relativistic and non-relativistic cases will be discussed. The dispersion relations are, respectively:

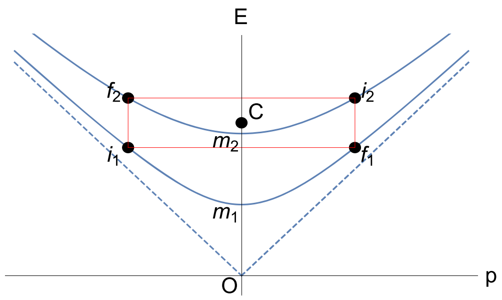

Figure 1. Kinematic solution of a relativistic one-dimensional elastic collision in the centre of mass frame. The initial momentum and energy of the colliding bodies are represented by i_1, i_2 and the final ones by f_1, f_2. The point C is the centre of the rectangle connecting them. The two hyperbolas, parameterised by the masses m_1 and m_2 correspond to the dispersion relation of (1).

$$E^2 = p^2 + m^2 \qquad \text{or} \qquad E = \text{const} + \frac{p^2}{2m}\,. \tag{1}$$

The velocity is given by $v = \mathrm{d}E/\mathrm{d}p$. The additive constant in the non-relativistic expression is immaterial, since only energy differences and not the absolute value of energy are important in that case. Thus, it can be arbitrarily chosen and the most natural choice when discussing non-relativistic physics is to set it to zero. However, if one thinks of that expression as the small velocity limit $v \ll c$ or $p \ll m$ of the relativistic one, then the constant is identified with the rest mass m. This will be the choice made below when plotting figure 4.

The expressions in (1) correspond, in the p–E plane, to families of hyperbolas and parabolas, both conic sections, parameterised by the mass m. The study of conic sections was introduced in ancient Greece by Menaechmus, developed by Aristaeus the Elder, Euclid and Archimedes, and stunningly extended and formalised by Apollonius of Perga [11].

Consider the elastic collision of two incoming bodies of masses m_1 and m_2 with some given momentum and energy. By definition, the outcome consists of the same bodies with different individual p and E, but preserving the total momentum and energy. Surely, the outgoing values of p and E can be found algebraically from (1), but the goal here is to find them with a ruler-and-compass construction in the p–E plane. In the centre of mass frame (vanishing total momentum), this is rather trivial since for each particle one simply takes $p \to -p$, see figure 1.

In order to address the case of a general inertial reference frame, notice that a boost of the situation of figure 1 must produce a parallelogram with the same area as the rectangle of the figure. This is a direct consequence of the properties of the Lorentz transformation, which preserves parallelism and areas in the p–E plane. This fact, together with the geometrical peculiarities of the hyperbola lead to the simple solution depicted in figure 2.

Given the hyperbolas and the initial points i_1, i_2, the solution is found in two steps:

(i) Find C as the middle point between i_1 and i_2.
(ii) Draw parallel lines to the OC segment from i_1 and i_2. The momentum and energy of the outgoing bodies, represented by f_1 and f_2 are the intersection of these lines with the hyperbolas.

A geometrical proof of this construction relies on a non-trivial feature of hyperbolas discovered by the ancient Greeks. Energy-momentum conservation requires that $i_1 f_2 i_2 f_1$ is a

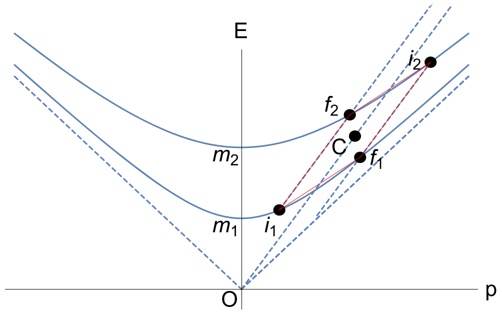

Figure 2. Kinematic solution of a relativistic one-dimensional elastic collision in an arbitrary inertial frame. The initial momentum and energy of the colliding bodies are represented by i_1, i_2 and the final ones by f_1, f_2. The point C is the centre of the parallelogram connecting them.

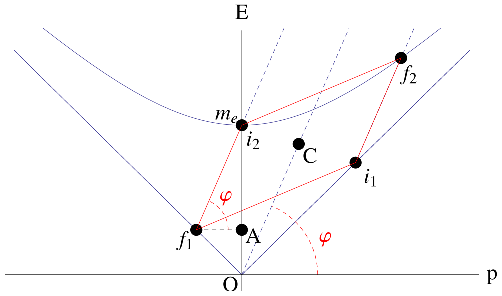

Figure 3. Kinematic solution of an example of one-dimensional Compton scattering. The auxiliary point A is placed at $p_A = 0$, $E_A = |p_{f1}|$.

parallelogram since only in that case C is the midpoint of both diagonals. Assuming the parallelogram exists, it can be shown that i_1f_2 and f_1i_2 are parallel to OC as follows. It was known to Apollonius of Perga that a straight line connecting the midpoints of two parallel chords of a hyperbola is a diameter [11], i.e. it passes through the centre, named O in figure 2. Since two hyperbolas of different m are simply rescalings of each other in the p–E plane, it immediately follows that the statement can be generalised to parallel chords of any hyperbolas defined by (1). This assertion implies that the OC line cuts the f_2i_2 and i_1f_1 segments at their midpoints. Clearly, the other sides of the parallelogram are parallel to the line connecting these midpoints, a fact that fully justifies the ruler-and-compass method described above.

It is worth pointing out that an interesting alternative geometrical solution of the same problem was presented by Bokor [12]. The technique described here is simpler, in the sense that it does not use auxiliary hyperbolas and therefore it can be carried out with ruler and compass on a piece of paper where the hyperbolas of (1) are drawn.

An interesting application of the procedure is the kinematical solution providing the one-dimensional version of the Compton shift. Consider a photon with momentum p_{i1} and energy E_{i1} impinging on a free electron at rest ($p_{i2} = 0$, $E_{i2} = m_e$). The photon is backscattered and the electron takes part of its initial kinetic energy. The graphical solution is depicted in figure 3. Notice that, since the photon is massless, the hyperbola degenerates into two straight lines $E = |p|$.

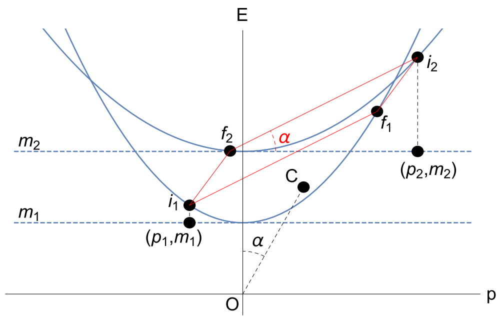

Figure 4. Kinematic solution of a non-relativistic one-dimensional elastic collision. The initial momentum and energy of the colliding bodies are represented by i_1, i_2 and the final ones by f_1, f_2. The point C corresponds to $\left(\frac{1}{2}(p_1 + p_2), \frac{1}{2}(m_1 + m_2)\right)$. The two parabolas, parameterised by the masses m_1 and m_2 correspond to the dispersion relation of (1), where const $= m$ has been chosen.

It is possible to extract the shift in the wavelength of the photon by using simple trigonometric arguments in the figure. By construction, the $f_1 i_2$ segment is parallel to OC and, therefore, the two angles marked as φ have the same value. Thus:

$$\tan\varphi = \frac{m_e - |p_{f1}|}{|p_{f1}|} = \frac{|p_{i1}| + m_e}{|p_{i1}|}, \tag{2}$$

where the first equality stems from the $f_1 A i_2$ triangle and the second one uses that C is the midpoint between i_1 and i_2. Then, $|p_{f1}| = m_e p_{i1}/(m_e + 2p_{i1})$ and, inserting the relation between wavelength and momentum of a photon $|p| = h/\lambda$, the solution yields

$$\lambda_f - \lambda_i = \frac{2h}{m_e} \tag{3}$$

which is the well-known Compton formula $\Delta\lambda = \frac{h}{m}(1 - \cos\theta)$ for backwards scattering ($\theta = \pi$).

Let us now turn to the one-dimensional collision in the non-relativistic case. It is not necessary to draw the parabolas to find a graphical solution [13]. However, it is illustrative to work out the kinematics in terms of a parallelogram similar to that of figure 2. The corresponding ruler-and-compass construction is depicted in figure 4.

Given the parabolas and the initial points i_1, i_2, the solution is found in two steps:

(i) Find C as the middle point between (p_1, m_1) and (p_2, m_2). Beware that none of these points lie on the parabolas.
(ii) Transport the angle α between the vertical and the OC segment and draw lines with angle α with respect to the horizontal that pass through the i_1, i_2 points. The resulting figure is a parallelogram as in the relativistic case, and the solution is found where each of the lines intersects the same parabola.

It is rather straightforward, although somewhat cumbersome, to demonstrate algebraically that the geometrical solutions described in this section are indeed correct. An appendix is provided outlining the proofs.

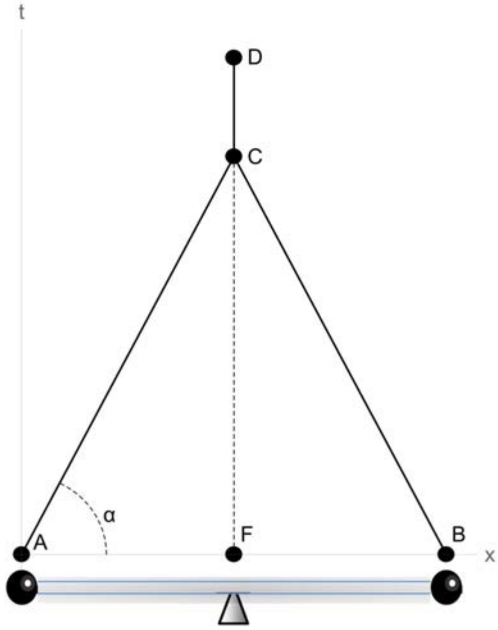

Figure 5. Fully inelastic collision of two bodies of equal mass in the centre of mass frame. The particles are initially placed at spacetime points A ($t = x = 0$) and B ($t = 0$, $x = L$), with initial velocities $v = \cot\ \alpha$ and $-v$. The values used in the plot are $v = 0.5$, $L = 1$. Due to the symmetry of the setup, after the collision at C ($t = L/(2v)$, $x = L/2$), the outgoing trajectory is of constant x. Point D is depicted at an arbitrary time after de collision D ($t = L/(2v) + t_d$, $x = L/2$) Projecting the trajectory, the point F ($t = 0$, $x = L/2$) is found, and it is identified with the fulcrum of a lever.

Similar methods to those presented for elastic one-dimensional cases can be applied to more general inelastic and/or two-dimensional problems. In two dimensions, hyperbolas and parabolas are substituted by hyperboloids and paraboloids (see also [12]).

3. Einstein's question and Archimedes' law of the lever

One of the most paradigmatic discoveries of Archimedes is the law of the lever, which he derived from geometrical reasoning [14]. It states that the quotient of the forces on both sides of a lever is equal to the inverse quotient of the length of the arms, from the fulcrum to the point where the force is applied.

$$\frac{F_{\mathrm{A}}}{F_{\mathrm{B}}} = \frac{l_{\mathrm{B}}}{l_{\mathrm{A}}}. \tag{4}$$

An application of it is the Roman steelyard: knowing a given weight and the position of the fulcrum at equilibrium, the value of a second weight can be deduced. In this section, this principle is used within spacetime diagrams in order to address Einstein's famous question [15]: *Does the inertia of a body depend upon its energy content?* Or, stated differently, does the mass of a body depend on its state of motion?

The starting point is a fully inelastic symmetric collision in the centre of mass frame, see figure 5. The left–right symmetry of the problem automatically guarantees that the outgoing

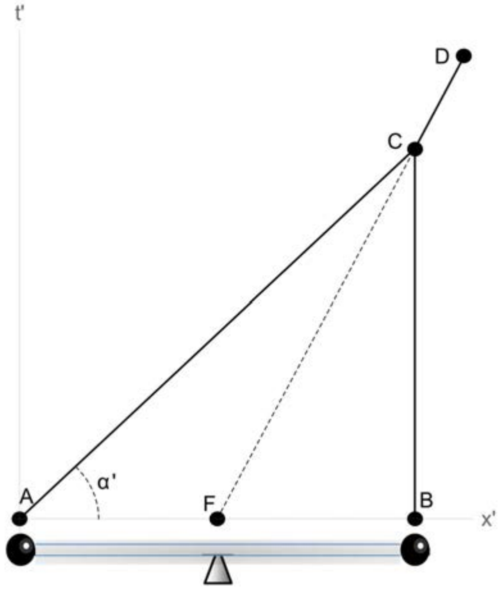

Figure 6. Non-relativistic Galilean boost of the situation of figure 5. The fulcrum remains in the centre of the lever, implying that the mass does not change due to motion of the body.

body is at rest, and that the initial bodies had the same inertia, irrespective of whether it depends on their velocity or not.

The solution of figure 5 applies to both the relativistic and non-relativistic cases. In order to emphasise the analogy with a lever, one has been depicted at the bottom of the figure. The fulcrum, coincident with the centre of mass at $t = 0$, is in the middle point. This is a visualisation of the equality of masses.

In order to address Einstein's question, one can study the process of figure 5 from the point of view of a different inertial frame. For convenience, the frame is chosen introducing a boost of velocity $-v$, such that the body initially placed at B is at rest.

The non-relativistic Galilean transformation, that leaves time invariant, can be written as

$$t' = t, \qquad x' = (x + vt). \tag{5}$$

Using these expressions, it is straightforward to find the graph of figure 6 and to check that the fulcrum F remains at the midpoint of the AB segment. This means that, in Galilean spacetime, the answer to Einstein's question is NO.

The situation taking into account the principles of special relativity is different. A Lorentz transformation, which leaves the speed of light invariant rather than time, reads, for a boost of velocity $-v$:

$$t' = \gamma(t + vx), \qquad x' = \gamma(x + vt), \tag{6}$$

with $\gamma = (1 - v^2)^{-\frac{1}{2}}$. The situation of figure 5 is transformed into the one of figure 7. The coordinates of points A, B, C, D and F in the x'–t' plane are computed from those of figure 5 using equation (6). They are

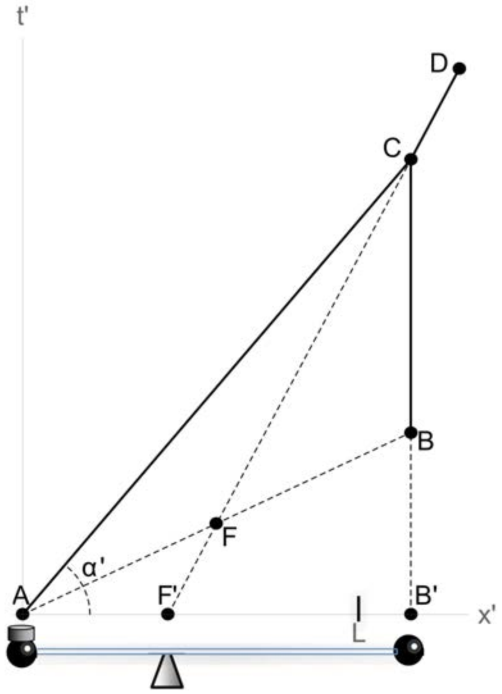

Figure 7. Relativistic Lorentz boost of the situation depicted in figure 5. F remains the midpoint between A and B but the projection F′ is closer to A than to B′. The fulcrum is not at the centre, implying that the mass of a moving body is larger than the mass of the same body at rest. This is pictorially represented by the disk putting an extra weight on the left arm of the lever. The point $x' = L$ has been displayed in order to emphasise that the length AB′ is different from AB in figures 5 and 6. The numerical values used in the figure for illustration are given in the caption of figure 5.

$$\begin{aligned}
&\mathrm{A}: (t' = 0,\, x' = 0),\\
&\mathrm{B}: (t' = \gamma v L,\, x' = \gamma L),\\
&\mathrm{C}: \left(t' = \frac{1}{2}\gamma L(v + v^{-1}),\, x' = \gamma L\right),\\
&\mathrm{D}: (t' = \gamma[L(v + v^{-1})/2 + t_d],\, x' = \gamma v(L + t_d)),\\
&\mathrm{F}: (t' = \gamma v L/2,\, x' = \gamma L/2).
\end{aligned} \tag{7}$$

Notice that the velocity of the boost $-v$ is chosen taking into account the velocities $|v|$ of the bodies in the centre of mass frame. The point F is still the midpoint of the AB segment. However, in order to compare with the lever, it is necessary to consider points at equal time t'. The line connecting F, C and D is parameterised by $x' = L/(2\gamma) + vt'$. The projection F′ (centre of mass at $t' = 0$), is the $t' = 0$ point of this line:

$$\mathrm{F}'\colon (t' = 0,\, x' = L/(2\gamma)). \tag{8}$$

Similarly, B′ is the $t' = 0$ point of the line connecting B and C:

$$\mathrm{B}'\colon (t' = 0,\, x' = \gamma L). \tag{9}$$

Clearly, F′ is displaced to the left with respect to the midpoint of the AB′ segment. The plot shows, unequivocally, that in Minkowskian spacetime the answer to Einstein's question is

YES. Moreover, it is clear that the inertial mass of a moving body is larger than the one of the body at rest.

In order to check that the analogy also holds at the quantitative level, it is necessary to compute the quotient of the lengths of the arms of the lever. From equations (7)–(9), it follows that the length of the AF′ segment is $\gamma L(1-v^2)/2$ and that the length of F′B′ is $\gamma L(1+v^2)/2$.

Thus, using Archimedes' law of the lever (4), and assuming that the gravitational force is proportional to the mass, the quotient of masses is

$$\frac{m_w}{m} = \frac{1+v^2}{1-v^2}, \tag{10}$$

where m_w is the mass of the body initially placed at A, which has rest mass m and velocity $w = \cot\alpha'$. From the coordinates of C in the boosted frame, equation (7), it is immediate to find out that $w = 2v/(1+v^2)$. This expression, together with (10), leads to

$$\frac{m_w}{m} = \frac{1}{\sqrt{1-w^2}}. \tag{11}$$

Thus, the well-known relativistic mass formula follows from Archimedes' law of the lever applied in a Minkowskian spacetime, which is built to accommodate Lorentz transformations. In modern language, one may say that the Archimedes' lever principle states that the fulcrum is at the position of the centre of mass. In the absence of external forces, the centre of mass undergoes uniform motion in a straight line, irrespectively of the inertial frame used for the description. Consequently, it is immediate to deduce that F′ is the centre of mass of the system at $t' = 0$ and (11) follows from simple trigonometry.

4. Final remarks

The main motivation of the present work is the development of novel tools to visually explain arid concepts of special relativity at different educational levels, including secondary school and basic undergraduate courses. It follows the strategy of a number of previous works, e.g. [6, 16], which explore the possibility of supplementing or substituting the usual algebraic formulation of special relativity with graphical constructions.

The kinematics of elastic collisions is governed by linear momentum and energy conservation laws plus the dispersion relations. The system can be solved with geometrical techniques relying on the properties of conic sections. For students, the transformation of seemingly abstruse algebraic relations into simple graphical constructions provides an alternative view and can be enlightening. The solution in terms of parallelograms, see figures 1–4, is illustrative also of the behaviour of energy and momentum under Lorentz transformations. In section 3, it was shown that geometrical reasoning for inelastic collisions is also useful when conveying far from intuitive concepts such as the equivalence between mass and energy (the remarkable history of this equivalence was described in [17]). In this sense, the simple connection made in section 3 makes crystal clear that paradigmatic properties of special relativity such as spacetime transformations and velocity dependent masses are intertwined. It is natural to conjecture that understanding this point can help students finding these notions for the first time in developing a global intuition of the underlying concepts. Furthermore, comparing the graphs for relativistic and non-relativistic cases allows the reader to infer the similarities and differences between both scenarios.

It should be mentioned that our goal here is not to describe all possible situations covered by our kind of approach, but to emphasise that this general framework allows one to develop new tools and possibilities. Classical geometry certainly underlies many aspects of modern physics and this fact may well be further exploited with educational purposes.

Visual tools as those described here, and simple physical devices useful for their illustration, have been used by one of the authors in secondary school courses [18]. The experience seems to bear out the expectation of an improved understanding of the underlying concepts by the students [19]. Certainly, this matter deserves further investigations from the perspective of physics education research.

This contribution is also aimed at describing some analogies and establishing an open dialogue between ancient Greek constructions, broadly represented by Archimedes, and modern science, symbolised by Einstein. The goal would have been achieved if the reader recognises common aesthetic patterns running from the classics to the contemporary and back.

Acknowledgments

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Appendix. Algebraic proofs of the graphical solutions

In this appendix, details of the algebraic proofs of the geometric solutions of figures 2 and 4 are spelled out.

The initial data for the relativistic collision (figure 2) are p_{i1}, E_{i1}, $_{pi2}$, E_{i2}, which are subject to the first expression of equation (1). The goal is to compute p_{f1}, E_{f1}, p_{f2}, E_{f2} by rewriting the geometrical procedure of section 2 in algebraic terms First of all, the bodies 1 and 2 preserve their mass in the collision and remain in the same hyperbola, resulting in

$$\begin{aligned} E_{i1}^2 - p_{i1}^2 &= E_{f1}^2 - p_{f1}^2 \\ E_{i2}^2 - p_{i2}^2 &= E_{f2}^2 - p_{f2}^2. \end{aligned} \tag{12}$$

Second, we require that the OC, $i_1 f_2$, $f_1 i_2$ segments are parallel, namely:

$$\frac{E_{\mathrm{C}}}{p_{\mathrm{C}}} = \frac{E_{f2} - E_{i1}}{p_{f2} - p_{i1}} = \frac{E_{i2} - E_{f1}}{p_{i2} - p_{f1}}. \tag{13}$$

Taking into account that C is the middle point between i_1 and i_2, implying $E_{\mathrm{C}} = \frac{1}{2}(E_{i1} + E_{i2})$ and $p_{\mathrm{C}} = \frac{1}{2}(p_{i1} + p_{i2})$, equation (13) can be rewritten as

$$\begin{aligned} (E_{f2} - E_{i1})(p_{i1} + p_{i2}) &= (E_{i1} + E_{i2})(p_{f2} - p_{i1}) \\ (E_{i2} - E_{f1})(p_{i1} + p_{i2}) &= (E_{i1} + E_{i2})(p_{i2} - p_{f1}). \end{aligned} \tag{14}$$

The system (12), (14) has the following solution:

$$
\begin{aligned}
p_{f1} &= \frac{p_{i1}((p_{i1}+p_{i2})^2 - E_{i1}^2 + E_{i2}^2) - 2p_{i2}E_{i1}(E_{i1}+E_{i2})}{(p_{i1}+p_{i2})^2 - (E_{i1}+E_{i2})^2}\\
p_{f2} &= \frac{p_{i2}((p_{i1}+p_{i2})^2 + E_{i1}^2 - E_{i2}^2) - 2p_{i1}E_{i2}(E_{i1}+E_{i2})}{(p_{i1}+p_{i2})^2 - (E_{i1}+E_{i2})^2}\\
E_{f1} &= \frac{E_{i1}((E_{i1}+E_{i2})^2 - p_{i1}^2 + p_{i2}^2) - 2p_{i1}E_{i2}(p_{i1}+p_{i2})}{-(p_{i1}+p_{i2})^2 + (E_{i1}+E_{i2})^2}\\
E_{f2} &= \frac{E_{i2}((E_{i1}+E_{i2})^2 + p_{i1}^2 - p_{i2}^2) - 2p_{i2}E_{i1}(p_{i1}+p_{i2})}{-(p_{i1}+p_{i2})^2 + (E_{i1}+E_{i2})^2}.
\end{aligned}
$$

It is immediate to check that these values provide a solution of the problem, since momentum and energy are conserved:

$$
p_{i1} + p_{i2} = p_{f1} + p_{f2}, \quad E_{i1} + E_{i2} = E_{f1} + E_{f2}. \tag{15}
$$

Moreover, as the geometric solution suggests, the $f_2 i_2$ segment is parallel to $i_1 f_1$:

$$
\frac{E_{i2} - E_{f2}}{p_{i2} - p_{f2}} = \frac{E_{f1} - E_{i1}}{p_{f1} - p_{i1}}. \tag{16}
$$

A similar check can be performed for the non-relativistic solution of figure 4. The initial and final points lie on their respective parabolas:

$$
\begin{aligned}
E_{i1} &= m_1 + \frac{p_{i1}^2}{2m_1^2}, \qquad E_{i2} = m_2 + \frac{p_{i2}^2}{2m_2^2}\\
E_{f1} &= m_1 + \frac{p_{f1}^2}{2m_1^2}, \qquad E_{f2} = m_2 + \frac{p_{f2}^2}{2m_2^2}\ .
\end{aligned} \tag{17}
$$

On the other hand, the segments $i_1 f_1, f_2 i_2$ are parallel, with an slope inverse to that of the OC line:

$$
\frac{E_{f1} - E_{i1}}{p_{f1} - p_{i1}} = \frac{E_{i2} - E_{f2}}{p_{i2} - p_{f2}} = \frac{p_{i1} + p_{i2}}{m_1 + m_2}, \tag{18}
$$

From equations (17), (18), one can compute the final energy and momenta:

$$
\begin{aligned}
p_{f1} &= \frac{p_{i1}(m_1 - m_2) + 2m_1 p_{i2}}{m_1 + m_2}\\
p_{f2} &= \frac{p_{i2}(m_2 - m_1) + 2m_2 p_{i1}}{m_1 + m_2}\\
E_{f1} &= (2m_1(m_1 + m_2)^2)^{-1}[2m_1^2((m_1 + m_2)^2 + 2p_{i2}^2)\\
&\quad + (m_1 - m_2)^2 p_{i1}^2 + 4m_1(m_1 - m_2)p_{i1}p_{i2})]\\
E_{f2} &= (2m_2(m_1 + m_2)^2)^{-1}[2m_2^2((m_1 + m_2)^2 + 2p_{i1}^2)\\
&\quad + (m_1 - m_2)^2 p_{i2}^2 - 4m_2(m_1 - m_2)p_{i1}p_{i2})].
\end{aligned} \tag{19}
$$

The solution given in equation (19) satisfies conservation of momentum and energy, and implies that the $i_1 f_2$ and $f_1 i_2$ segments are parallel.

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