Geometric Power and Poynting Vector: a Physical Derivation for Harmonic Power Flow using Geometric Algebra

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***Abstract*—This document aims to establish an alternative physical formulation for the harmonic power flow in electrical systems provided by Geometric Algebra (GA) and the Poynting Vector (PV) and Poynting Theorem (PT). Given the traditional definition of PV (Abraham approach) as the vector product of the electric field and magnetic field, we exploit the property of the vector product as a dual form of the much more powerful wedge product operator from exterior algebra. Using concepts of vector spaces, we develop a completely GA-based approach founded on top of the isomorphism among periodic time-domain signals and Euclidean spaces. Our investigations shed more light on the long-running discussion of electric power flow in non-sinusoidal and non-linear electrical power systems.**

***Index Terms*—geometric algebra, geometric power, harmonic power, Poynting vector**

I. Introduction

The Poynting Theorem (PT) establishes the conservation of energy for the electromagnetic field and is derived from Maxwell's equations and Lorentz law. For a linear media and ohmic conductor, the integral form is usually written as

$$-\oiint_{\partial V} \vec{S}\cdot d\vec{a} = \int_V \vec{J}\cdot\vec{E}dV + \frac{\partial}{\partial t}\int_V udV \qquad (1)$$

where V is a generic volume, ∂V is the boundary surface of that volume with unitary normal surface $d\vec{a}$, u is electromagnetic energy density, $\vec{E}$ is the electric field vector, $\vec{J}$ is the current density vector and $\vec{S} = \vec{E}\times\vec{H}$ is the PV. It is named after J.H. Poynting who first presented it in the late nineteenth century. Its physical interpretation has always been linked to the electromagnetic field local energy density flow. Its relevance is unquestionable when it comes to electromagnetic radiation. However, it has had limited use in power electrical circuits, where there is a predominant current conduction process (although the magnetic induction effect in transformers is completely driven by electromagnetic waves). Despite the efforts made by renowned researchers such as Emanuel [1] or DeLeon [2] to reconcile the flow of active and non-active power with PT's postulates, its use has not been widespread in the power engineering community. Interestingly enough, so far a single researcher [3] has criticized its validity, to the best knowledge of the authors. However, solid and well-founded proofs have been presented to refute these claims [4].

Furthermore, GA has been successfully applied during the last years to many scientific and engineering fields such as physics, robotics, computer vision, etc., being power systems one of them. GA framework has been used to define the Geometric Power (GP) [5]–[7] because it is extremely useful at capturing the inherent multidimensionality of time and frequency multi-phase circuits. Hitherto, its application has been certainly limited to the frequency domain and single-phase systems in the presence of harmonics through a $\mathcal{C}l_{2k,0,0}$ Clifford algebra where k is the number of harmonics in the voltage/current waveform (if *dc* is present, then $2k+1$ dimensions will be needed). In the formalism of GA, $\mathcal{C}l_{p,q,r}$ means that the basis contains p vectors squaring to $+1$, q vectors squaring to 0 and r vectors squaring to -1. For example, Space-Time Algebra (STA) makes use of a Minkowski space with $\mathcal{C}l_{3,0,1}$. Recently, the GA approach has been extended to multi-phase circuits in time [8] and frequency domain [9].

Therefore, we are interested in devising a foundational connection between the PT (formulated in GA terminology) and the above-mentioned Geometric Power that contribute to a better understanding of harmonic power flow under nonsinusoidal conditions and non-linear loads. In this paper, we present a GA-based version of the PT, from which we derive the proposed GP in [7]. This proposal is unique and cannot be addressed through the algebra of complex numbers in the presence of multiple harmonics. We highlight that the term *harmonic power flow* not only refers to the traditional accepted concept associated to a ficticious power obtained by considering armonics of voltage and current of the same

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frequency, but we also extend the formulation to a new ficticious power flow originated by the interaction of mixed frequency voltage and current components.

II. Geometric Algebra applied to power systems

Starting from the pioneering work of Menti [10], a refined mathematical framework was developed in [7] and [8] that can be applied to power systems under any supply or load condition. One of its most valuable merits is that both, time and frequency domain approaches, can be formulated. In this sense, GA reveals its prominent significance as a unified framework to describe physical processes. GA is pure math, so this framework should not be understood as a "yet another new power theory", but rather as a new, advanced, and unified way to look into the physics of the problem.

The way that GA applies to power systems in the frequency domain relies on the use of an orthonormal base

$$\boldsymbol{\sigma} = \{\boldsymbol{\sigma}_0, \boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \ldots, \boldsymbol{\sigma}_{2n}\} \tag{2}$$

defined for a vector space in $\mathbb{R}^{2n+1}$ that is isomorphic to the well-known Fourier basis for time domain periodic functions $\{1, \sqrt{2}\cos\omega t, \sqrt{2}\sin\omega t, \ldots, \sqrt{2}\cos n\omega t, \sqrt{2}\sin n\omega t\}$, where ω is a constant and n is the highest harmonic order. Any vector $\boldsymbol{v}$ can be represented as a linear combination of unitary basis vectors $\boldsymbol{\sigma}_k$:

$$\boldsymbol{v} = \sum_{k=0}^{2n} v_k \boldsymbol{\sigma}_k = v_0\boldsymbol{\sigma}_0 + v_1\boldsymbol{\sigma}_1 + \ldots + v_{2n}\boldsymbol{\sigma}_{2n} \tag{3}$$

Note that the term v_0 accounts for the dc term. From now on, this term will be deliberately ignored. Under this assumption, it is possible to establish a new algebra $\mathcal{G}^n$ with a bilinear form: the geometric product. For two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$, it can be defined as:

$$\boldsymbol{M} = \boldsymbol{u}\boldsymbol{v} = \boldsymbol{u}\cdot\boldsymbol{v} + \boldsymbol{u}\wedge\boldsymbol{v} \tag{4}$$

which can be seen as the sum of the traditional scalar or inner product plus the so-called wedge or Grassmann product. The latter fulfills the anti-commutativity property:

$$\boldsymbol{u}\wedge\boldsymbol{v} = -\boldsymbol{v}\wedge\boldsymbol{u} \tag{5}$$

The above entity is commonly known as *bivector* and is a new object not found previously in linear algebra. For example, a two dimensional vector $\boldsymbol{u} = u_1\boldsymbol{\sigma}_1 + u_2\boldsymbol{\sigma}_2$ and $\boldsymbol{v} = v_1\boldsymbol{\sigma}_1 + v_2\boldsymbol{\sigma}_2$, can be multiplied using the geometric product

$$\begin{aligned}\boldsymbol{M} = \boldsymbol{u}\boldsymbol{v} &= (u_1\boldsymbol{\sigma}_1 + u_2\boldsymbol{\sigma}_2)(v_1\boldsymbol{\sigma}_1 + v_2\boldsymbol{\sigma}_2) \\ &= \underbrace{(u_1v_1 + u_2v_2)}_{\langle\boldsymbol{M}\rangle_0} + \underbrace{(u_1v_2 - u_2v_1)}_{\langle\boldsymbol{M}\rangle_2}\boldsymbol{\sigma}_{12}\end{aligned} \tag{6}$$

where $\boldsymbol{M}$ consists of two elements, the term $\langle\boldsymbol{M}\rangle_0$ which is a scalar and the term $\langle\boldsymbol{M}\rangle_2$ which is a bivector. As these elements are of a different nature, $\boldsymbol{M}$ is commonly referred to as *multivector*. The operator $\langle\cdot\rangle_k$ refers to k-grade component of a multivector. The norm of a multivector is:

$$\|\boldsymbol{M}\| = \sqrt{\langle\boldsymbol{M}^\dagger\boldsymbol{M}\rangle_0} \tag{7}$$

where $\boldsymbol{M}^\dagger$ is the reverse of $\boldsymbol{M}$ (see [11] for details). Note the details of the chosen notation. In the GA domain, capital bold letters are used for bivectors (in general, for multivectors) and small bold letters for vectors (instead of the more traditional arrow or bar symbol).

A. Frequency Domain Geometric Power Definition

In this work, the frequency approach is chosen to explain the power flow. The main reason is the great interest in the power community to compute harmonic power flow completely in this domain. The rationale for a GA approach is that the transformation of periodic signals from the time to the geometric domain can reveal new insights for the harmonic power flow not previously disclosed in the phasor or complex domain. For this purpose, the isomorphic property among vector spaces is exploited as detailed in [7], and reproduced here for convenience:

$$\begin{array}{ccc} 1 & \longleftrightarrow & \boldsymbol{\sigma}_0 \\ \sqrt{2}\cos\omega t & \longleftrightarrow & \boldsymbol{\sigma}_1 \\ \sqrt{2}\sin\omega t & \longleftrightarrow & \boldsymbol{\sigma}_2 \\ & \vdots & \\ \sqrt{2}\cos n\omega t & \longleftrightarrow & \boldsymbol{\sigma}_{2n-1} \\ \sqrt{2}\sin n\omega t & \longleftrightarrow & \boldsymbol{\sigma}_{2n} \end{array} \tag{8}$$

Based on (8), a general voltage (or current) waveform is then transformed as

$$\boldsymbol{u} = U_0\boldsymbol{\sigma}_0 + \sum_{k=1}^{n} U_{kc}\boldsymbol{\sigma}_{2k-1} + U_{ks}\boldsymbol{\sigma}_{2k} \tag{9}$$

where $U_{kc} = U_k\cos\varphi_k$ and $U_{ks} = U_k\sin\varphi_k$. The same transformation can be applied to $i(t)$ in order to calculate the geometric current $\boldsymbol{i}$. It is worth noting that $\boldsymbol{i}$ may include harmonics not present in the voltage. See [7] for further details about norms and current decomposition. For the simple case of a sinusoidal linear single-phase circuit, a generic voltage is $v(t) = \sqrt{2}(v_c\cos\omega t + v_s\sin\omega t)$, while the current is $i(t) = \sqrt{2}(i_c\cos\omega t + i_s\sin\omega t)$. In this case, the geometric voltage is $\boldsymbol{v} = v_c\boldsymbol{\sigma}_1 + v_s\boldsymbol{\sigma}_2$ and the geometric current $\boldsymbol{i} = i_c\boldsymbol{\sigma}_1 + i_s\boldsymbol{\sigma}_2$. The geometric power can be computed as showed in (6),

$$\boldsymbol{M} = \boldsymbol{v}\boldsymbol{i} = \underbrace{(v_ci_c + v_si_s)}_{M_p=P} + \underbrace{(v_ci_s - i_cv_s)}_{M_q=Q}\boldsymbol{\sigma}_{12} \tag{10}$$

As expected, $\boldsymbol{M}$ is composed of a scalar (the active power P) and a bivector (the reactive power Q) term. Note that they are obtained from interactions of the same frequency components. Let's now assume a non-linear load that generates a harmonic current like $i_h(t) = \sqrt{2}(i_c\cos 2\omega t + i_s\sin 2\omega t)$, so the geometric current is $\boldsymbol{i}_h = i_c\boldsymbol{\sigma}_3 + i_s\boldsymbol{\sigma}_4$. The geometric power is now

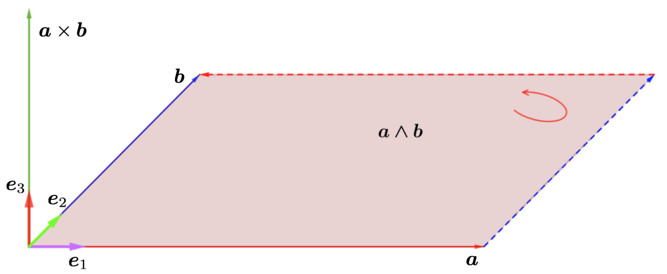

Figure 1. Cross (green vector) and wedge (red plane) product of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$.

$$\boldsymbol{M}_h = v_c i_c \boldsymbol{\sigma}_{13} + v_c i_s \boldsymbol{\sigma}_{14} + v_s i_c \boldsymbol{\sigma}_{23} + v_s i_s \boldsymbol{\sigma}_{24} \tag{11}$$

The terms in $\boldsymbol{M}_h$ are all bivectors. They are the result of the interaction of components of different frequencies. Note also, that whenever the bivector part is present, the efficiency of the delivery system is not optimized, so a current decomposition procedure could be carried out to cancel that part of the current that accounts for the non-active part of the geometric power. This non-active power is always related to the wedge product of voltage and current $\boldsymbol{v} \wedge \boldsymbol{i}$. It follows that the geometric power $\boldsymbol{M}$ can capture the non-active cross-product power terms. This is not the case for the traditional complex apparent power.

III. Electromagnetic fields formulation using Geometric Algebra

Geometric Algebra applied to electromagnetism was developed by D. Hestenes [11] in the 1970s. The underlying idea is that, unlike the electric field $\vec{E}$, the magnetic field $\vec{H}$ (or $\vec{B}$) is not a polar but an axial vector resulting from the vector product of two polar vectors. A typical case where an axial vector is obtained is the computation of the magnetic field caused by a wire carrying a current. According to the classical approach (Biot-Savart's law), the result for a 3D case is

$$d\vec{B} = \frac{\mu I}{4\pi} \frac{d\vec{l} \times \vec{r}}{r^3} \tag{12}$$

where I is the electric current, $d\vec{l}$ is an infinitesimal part of the wire and $\vec{r}$ is the vector position.

As depicted in figure 1, the cross product $\boldsymbol{a} \times \boldsymbol{b}$ is just the vector normal to the plane defined by the bivector $\boldsymbol{a} \wedge \boldsymbol{b}$. It is worth mentioning that this property is only satisfied in 3D. Axial vectors are a dual (but limited) representation of a more general concept: bivectors (see reference [12] for a comprehensive list of the benefits of using bivectors instead of axial vectors). Therefore, it is claimed that the *magnetic field is a bivector quantity in its essence*. Thus, the bivector representation for the magnetic induction is as follows

$$d\boldsymbol{B} = \frac{\mu I}{4\pi} \frac{d\boldsymbol{l} \wedge \boldsymbol{r}}{r^3} \tag{13}$$

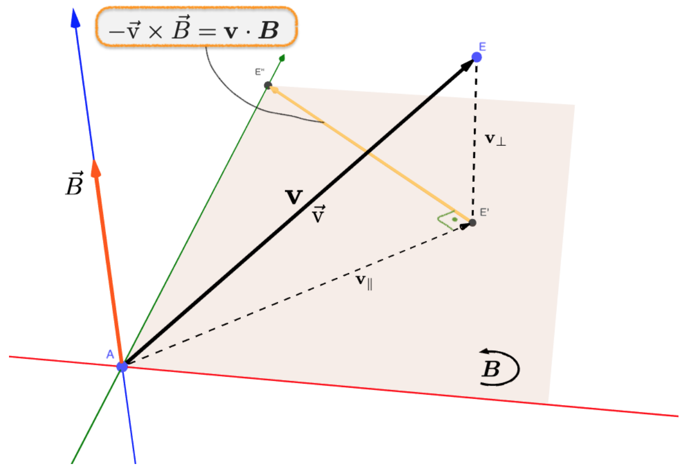

Figure 2. Projection (dot product) of a vector $\mathbf{v}$ onto a bivector $\boldsymbol{B}$. The result is the opposite to the cross product of the vector $\vec{v} = \mathbf{v}$ and vector $\vec{B}$ (dual of the bivector $\boldsymbol{B}$).

Note again the use of bold and lower case for vectors and upper case for bivectors or multivectors in general. The GA approach for electromagnetic fields leads to a new representation of the Lorentz force

$$\boldsymbol{f} = q\boldsymbol{e} + q\boldsymbol{B} \cdot \mathbf{v} \tag{14}$$

where $\boldsymbol{e}$ is the electric field vector, $\mathbf{v}$ is the velocity vector of a charged particle and $\boldsymbol{B}$ is the magnetic induction bivector. Note that the property $\vec{v} \times \vec{B} = -\mathbf{v} \cdot \boldsymbol{B} = \boldsymbol{B} \cdot \mathbf{v}$ has been used in (14). For clarity, figure 2 shows a geometric representation for this property.

Once we have highlighted the different nature of the electric and magnetic induction fields in the GA domain, it is interesting to point out the new form for the Poynting Vector. Note that from now on, the Abraham approach [13] will be followed, i.e., the magnetic bivector field $\boldsymbol{H}$ will be used instead of the induction bivector $\boldsymbol{B}$ to compute the PV

$$\boldsymbol{s} = \boldsymbol{H} \cdot \boldsymbol{e} \tag{15}$$

The above expression is generally valid for any dimension and matches the classical one, i.e., $\boldsymbol{s} = \vec{E} \times \vec{H}$ in three dimensions.

IV. Poynting vector and geometric power

In the previous section, the GA approach for PV from a pure electromagnetism point of view has been introduced. However, the goal of the presented investigations is to shed light on the relationship between PV and the geometric power $\boldsymbol{M}$ proposed in [7]. Is there any way to derive $\boldsymbol{M}$ from PV? For this task, the vector field generated by the PV needs to be analyzed.

From a practical perspective, we are interested in transmission systems delivering energy/power from a source to a load in the real world, i.e., in a 3D space. However, a

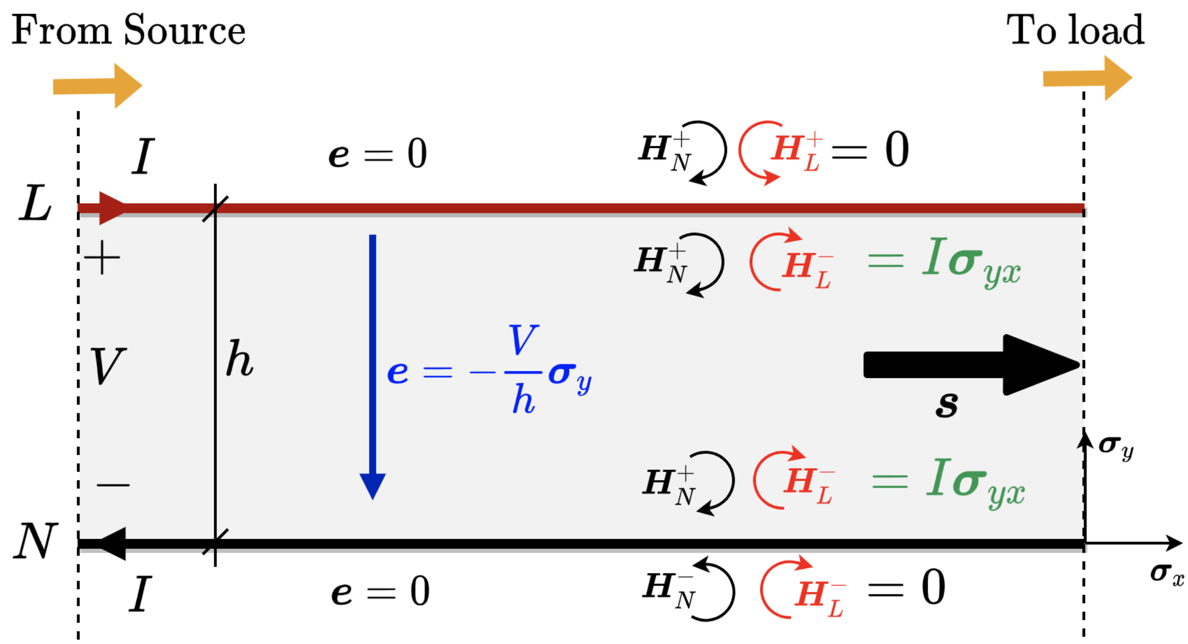

Figure 3. Electric vector $\boldsymbol{e}$, magnetic bivector $\boldsymbol{H}$ and Poynting vector $\boldsymbol{s}$ caused by two long and thin conductors carrying a current I in a 2D world.

new theoretical 2D physical space will be presented here for convenience and simplicity. Note that nothing prevents Maxwell's equations (from which circuit theory is derived) to be applied in two-dimensional flat worlds. It is outside of the current work a full justification of the above assertion, but the reader is encouraged to read references [14], [15]. Inmediately, it is obvious that a mathematical inconvenient arises if we use equation (12) to compute the magnetic field in a 2D world. The use of the vector product is not even defined in such a space. This is a major and serious drawback of vector calculus. In contrast, the use of the wedge product does not pose any problem since the result is a bivector within the original space. Moreover, the generalization to the real 3D space is also straightforward [12], [16].

To emphasize the benefits of GA and without loss of generality, a planar (flat) circuit with two long and thin conductors (L and N) separated by a distance h and connecting a power source and a load is presented in figure 3. A new vector basis $\boldsymbol{\sigma}' = \{\boldsymbol{\sigma}_x, \boldsymbol{\sigma}_y\}$, different from basis $\boldsymbol{\sigma}$ in (2), is used to describe the Euclidean geometry in this planar 2D space. There are two regions of interest: the one between the conductors (internal) and the remainder (external). According to [14], the field calculations can be performed exclusively in 2D using GA. In the internal region, the resulting electric and magnetic fields are constant, leading also to a constant PV. In contrast, in the external region, all the fields are zero, so we can conclude that the power flow is confined through the internal region.

The field values for the internal region are

$$
\begin{aligned}
\boldsymbol{e} &= -\frac{V}{h}\boldsymbol{\sigma}_y \\
\boldsymbol{H}_N^+ = \boldsymbol{H}_L^- &= \frac{I}{2}\boldsymbol{\sigma}_{yx} \\
\boldsymbol{H}_N^- = \boldsymbol{H}_L^+ &= \frac{I}{2}\boldsymbol{\sigma}_{xy} \\
\boldsymbol{H} = \boldsymbol{H}_N^+ + \boldsymbol{H}_L^- &= I\boldsymbol{\sigma}_{yx}
\end{aligned} \tag{16}
$$

where $\boldsymbol{H}_L^+$ and $\boldsymbol{H}_N^+$ are the magnetic field in the upper part of the wire L and N, respectively. Similarly, $\boldsymbol{H}_L^-$ and $\boldsymbol{H}_N^-$ are the magnetic field at the bottom part of the wires. In this way, the PV is computed according to (15) as

$$
\boldsymbol{s} = \boldsymbol{H} \cdot \boldsymbol{e} = \frac{1}{2}\left(\boldsymbol{H}\boldsymbol{e} - \boldsymbol{e}\boldsymbol{H}\right) = \frac{VI}{h}\boldsymbol{\sigma}_x \tag{17}
$$

Notice that the resultant PV is a vector parallel to the wires. The above implies that the vector $\boldsymbol{s}$ cannot contain components in the y axis. This result is already familiar and is in good agreement with the three-dimensional case for rectilinear conductors [1], [4], [17]. For an arbitrary voltage $v(t)$ and current $i(t)$ the result is similar but the PV can oscillate between positive and negative values in $\boldsymbol{\sigma}_x$ direction. In (17), there is clear evidence of the interaction among the electric and magnetic fields which plays an important role in the power derivation formulae. The application of (1) to a general voltage and current for the 2D case is straightforward

$$
p(t) = \oiint_{\partial V} \boldsymbol{s} \cdot d\boldsymbol{a} = \boldsymbol{s} \cdot h\boldsymbol{\sigma}_x = s(t)h = v(t)i(t) \tag{18}
$$

Notice that the use of GA for a 2D world gives the expected results for the instantaneous power using the PV in a completely general way under the quasi-static field condition [18]. Particularly, periodic signals are of great interest from an engineering point of view. The vector field $\boldsymbol{e}$ and bivector field $\boldsymbol{H}$ can be obtained as an infinite sum of time-harmonic components at any point in space

$$\begin{aligned}\boldsymbol{e}(t) &= e(t)\boldsymbol{\sigma}_y = \sum_{m\in M}\boldsymbol{e}_m(t) = \sum_{m\in M} e_m(t)\boldsymbol{\sigma}_y \\ \boldsymbol{H}(t) &= H(t)\boldsymbol{\sigma}_{xy} = \sum_{l\in L}\boldsymbol{H}_l(t) = \sum_{l\in L} H_l(t)\boldsymbol{\sigma}_{xy}\end{aligned} \tag{19}$$

where M and L are the set of harmonics in electric and magnetic field, respectively. Note that, in general, the set $M \neq L$. Therefore, substituting (19) into (17) and considering (18), the resulting PV is

$$\boldsymbol{s}(t) = e(t)H(t)\boldsymbol{\sigma}_x = \left(\sum_{m\in M} e_m(t)\right)\left(\sum_{l\in L} H_l(t)\right)\boldsymbol{\sigma}_x \tag{20}$$

and the instantaneous power is

$$p(t) = s(t)h = \left(\sum_{\substack{m\in M\\ l\in L}} e_m(t)H_l(t)\right) h \tag{21}$$

It can be observed that (21) leads to a result with a main implication: there are cross-frequency terms due to the interaction of harmonics of different frequencies between the electric and magnetic fields. This is an important aspect that can be completely captured by GA. It must be emphasized that this is not possible with complex algebra because of its inherent limitation in dimensions. The rationale for this reasoning is presented in [7].

The basis in (2) can be used to account for the multidimensionality of the harmonic components in a given signal $x(t)$ (we intentionally exclude the *dc* component for simplicity). Let us suppose a n-dimensional harmonic signal $x(t)$

$$x(t) = \sum_{k=1}^{n} x_k(t) = \sum_{k=1}^{n}\sqrt{2}X_k\cos\left(k\omega t + \varphi_k\right) \tag{22}$$

By using the isomorphic property of vector spaces [7], it can be transferred from the time to the geometric domain by means of the basis $\boldsymbol{\sigma}$ defined in (2)

$$\begin{aligned}\boldsymbol{x} &= \sum_{k=1}^{n} X_k\cos\varphi_k\boldsymbol{\sigma}_{2k-1} - X_k\sin\varphi_k\boldsymbol{\sigma}_{2k} \\ &= \bar{X}_1\boldsymbol{\sigma}_1 + \ldots + \bar{X}_{2n}\boldsymbol{\sigma}_{2n}\end{aligned} \tag{23}$$

where $\bar{X}_{2k-1} = X_k\cos\varphi_k$ and $\bar{X}_{2k} = -X_k\sin\varphi_k$. The difference between the basis $\boldsymbol{\sigma}'$, which represents the multidimensionality of the geometric space in which we live, and $\boldsymbol{\sigma}$, which represents the multidimensionality of the vector space of harmonic functions, is expressly emphasized. For a two time signals $x(t)$ and $y(t)$, $\boldsymbol{x}$ and $\boldsymbol{y}$ are the transformed vectors in GA, respectively. For simplicity, we'll assume that both signals have the same harmonic content. Using (6), the geometric product is

$$\begin{aligned}\boldsymbol{xy} &= \left(\bar{X}_1\boldsymbol{\sigma}_1 + \ldots + \bar{X}_{2n}\boldsymbol{\sigma}_{2n}\right)\left(\bar{Y}_1\boldsymbol{\sigma}_1 + \ldots + \bar{Y}_{2n}\boldsymbol{\sigma}_{2n}\right) \\ &= \sum_{k=1}^{2n}\bar{X}_k\bar{Y}_k + \sum_{\substack{l,k\\ l<k}}^{2n}\left(\bar{X}_l\bar{Y}_k - \bar{X}_k\bar{Y}_l\right)\boldsymbol{\sigma}_{lk}\end{aligned} \tag{24}$$

Note that there is a scalar part that accounts for the dot product and a bivector part that accounts for the wedge product. Now, we are in a position where the coordinate of PV in (17) can be transferred to the harmonic GA representation. The coordinates of the electric and magnetic field in (19) now becomes

$$\begin{aligned}\boldsymbol{e}_e &= \sum_{k=1}^{2n} e_k\boldsymbol{\sigma}_k = e_1\boldsymbol{\sigma}_1 + \ldots + e_{2n}\boldsymbol{\sigma}_{2n} \\ \boldsymbol{h}_H &= \sum_{k=1}^{2n} H_k\boldsymbol{\sigma}_k = H_1\boldsymbol{\sigma}_1 + \ldots + H_{2n}\boldsymbol{\sigma}_{2n}\end{aligned} \tag{25}$$

where $\boldsymbol{e}_e$ is the coordinate for the spatial electric vector $\boldsymbol{e}$ and $\boldsymbol{h}_H$ is the only spatial coordinate of the magnetic bivector $\boldsymbol{H}$. Note that both $\boldsymbol{e}_e$ and $\boldsymbol{h}_H$ are vectors. Following (24), the result is

$$\boldsymbol{e}_e\boldsymbol{h}_H = \sum_{k=1}^{2n} e_kH_k + \sum_{\substack{k,l\\ k<l}}^{2n}\left(e_kH_l - e_lH_k\right)\boldsymbol{\sigma}_{kl} \tag{26}$$

Recalling expression (18) and choosing the same boundary as in figure 3, the result for the right side of (18) is a scaled version of (26). Thus, the left hand of (18) can be also transferred to GA by virtue of the isomorphic property. We call it multiphase geometric power

$$\boldsymbol{M} = \boldsymbol{ui} \tag{27}$$

For the specific case of single-phase systems with multiple harmonics (for simplicty, we consider the same for voltage and current), the voltage vector is $\boldsymbol{u} = u_1\boldsymbol{\sigma}_1 + \ldots + u_{2n}\boldsymbol{\sigma}_{2n}$ and the voltage current is $\boldsymbol{i} = i_1\boldsymbol{\sigma}_1 + \ldots + i_{2n}\boldsymbol{\sigma}_{2n}$. The geometric power is

$$\boldsymbol{M} = \boldsymbol{ui} = \sum_{k=1}^{2n} u_ki_k + \sum_{\substack{k,l\\ k<l}}^{2n}\left(u_ki_l - u_li_k\right)\boldsymbol{\sigma}_{kl} \tag{28}$$

We can now compare term by term equations (26) and (28)

$$
\begin{aligned}
P &= \sum_{k=1}^{2n} u_k i_k = d \sum_{k=1}^{2n} e_k H_k \\
\boldsymbol{M}_n &= \sum_{\substack{k,l \\ k<l}}^{2n} \left(u_k i_l - u_l i_k\right) \boldsymbol{\sigma}_{kl} = h \sum_{\substack{k,l \\ k<l}}^{2n} \left(e_k H_l - e_l H_k\right) \boldsymbol{\sigma}_{kl}
\end{aligned}
\tag{29}
$$

where P is the active power, the result of the product among similar components and same frequency of the voltage and current. $\boldsymbol{M}_n$ is the non-active geometric power and is the result of the sum of two types of products: cross-frequency and in-quadrature terms among voltage and current (terms like $\boldsymbol{\sigma}_{(2k-1)(2k)}$). The coordinates of the latter are the well-known reactive power in the Budeanu sense. The scaling factor h accounts for the geometry of the enclosing surface (line) of the circuit.

It can be seen that the geometric power has a well-defined physical significance by means of the isomorphism among vector spaces. It is founded on PT and PV, which leads to the traditional concepts of active power, reactive power, and non-active power in the frequency domain.

V. Conclusions

An alternative physical formulation for the harmonic power flow in electrical systems provided by Geometric Algebra and the Poynting Vector has been presented in this paper. By using a new approach based on a simple world of reduced dimensionality (two dimensions), the PV has been formulated considering the magnetic field as a bivector rather than an axial vector. The scalar product of the bivector magnetic field and the vector electric field yields the GA counterpart for the PV. For the quasi-static condition in electrical circuits, it is obtained the traditional result of a vector parallel to the wires entering the circuit. Finally, the comparison term by term among PV and geometric power, confirms the physical foundation of the latter. For the active power, it is the result of the product of frequency-like terms among voltage and current, while for the reactive and non-active power it is the result of cross-frequency and in-quadrature terms among voltage and current.

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References

[1] A. E. Emanuel, Poynting vector and the physical meaning of nonactive powers, in: Proceedings of the 21st IEEE Instrumentation and Measurement Technology Conference (IEEE Cat. No. 04CH37510), Vol. 3, IEEE, 2004, pp. 1742–1747.

[2] F. De Léon, J. Cohen, AC power theory from poynting theorem: Accurate identification of instantaneous power components in nonlinear-switched circuits, IEEE Trans. on Pow. Del. 25 (4) (2010) 2104–2112.

[3] L. S. Czarnecki, Could power properties of three-phase systems be described in terms of the poynting vector?, IEEE Transactions on Power Delivery 21 (1) (2005) 339–344.

[4] A. E. Emanuel, About the rejection of poynting vector in power systems analysis, Electrical Power Quality and Utilisation. Journal 13 (1) (2007) 43–49.

[5] F. G. Montoya, Geometric algebra in nonsinusoidal power systems: A case of study for passive compensation, Symmetry 11 (10) (2019) 1287.

[6] F. G. Montoya, R. Baños, A. Alcayde, F. M. Arrabal-Campos, E. Viciana, Analysis of non-active power in non-sinusoidal circuits using geometric algebra, International Journal of Electrical Power & Energy Systems 116 (2020) 105541.

[7] F. G. Montoya, R. Baños, A. Alcayde, F. M. Arrabal-Campos, J. Roldán-Pérez, Vector geometric algebra in power systems: An updated formulation of apparent power under non-sinusoidal conditions, Mathematics 9 (11) (2021) 1295.

[8] F. G. Montoya, J. Roldán-Pérez, A. Alcayde, F. M. Arrabal-Campos, R. Banos, Geometric algebra power theory in time domain, arXiv preprint arXiv:2002.05458 (2020).

[9] F. G. Montoya, R. Baños, A. Alcayde, F. M. Arrabal-Campos, J. Roldán Pérez, Geometric algebra framework applied to symmetrical balanced three-phase systems for sinusoidal and non-sinusoidal voltage supply, Mathematics 9 (11) (2021) 1259.

[10] A. Menti, T. Zacharias, J. Milias-Argitis, Geometric algebra: a powerful tool for representing power under nonsinusoidal conditions, IEEE Transactions on Circuits and Systems I: Regular Papers 54 (3) (2007) 601–609.

[11] D. Hestenes, G. Sobczyk, Clifford algebra to geometric calculus: a unified language for mathematics and physics, Vol. 5, Springer Science & Business Media, 2012.

[12] B. Jancewicz, Multivectors and Clifford algebra in electrodynamics, World Scientific, 1989.

[13] R. Loudon, C. Baxter, Contributions of john henry poynting to the understanding of radiation pressure, Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences 468 (2143) (2012) 1825–1838.

[14] X. P. Orbán, J. Mira, Dimensional scaffolding of electromagnetism using geometric algebra, European Journal of Physics 42 (1) (2020) 015204.

[15] K. T. McDonald, Electrodynamics in 1 and 2 spatial dimensions, https://physics.princeton.edu/ mcdonald/examples/2dem.pdf, accessed: 2012-01-30 (2019).

[16] D. Hestenes, New foundations for classical mechanics, Vol. 15, Springer Science & Business Media, 2012.

[17] H. Grabinski, F. Wiznerowicz, Energy transfer on three-phase high-voltage lines: the strange behavior of the poynting vector, Electrical Engineering 92 (6) (2010) 203–214.

[18] R. Redlich, Note on power and poynting vector in low-frequency circuits, IEEE Trans. on Education 27 (2) (1984) 109.