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The bismar scale and elastic collisions: a geometrical analogy

Xabier Prado[1], **Angel Paredes**[2,*], **Iván Area**[2,3], **José Manuel Domínguez Castiñeiras**[1] and **Jorge Mira**[4]

[1] Departamento de Didáctica das Ciencias Experimentais, Facultade de Ciencias da Educación, Universidade de Santiago de Compostela, E-15782 Santiago de Compostela, Spain
[2] Instituto de Física e Ciencias Aeroespaciais (IFCAE), Universidade de Vigo, Campus As Lagoas, E-32004 Ourense, Spain
[3] Universidade de Vigo, Departamento de Matemática Aplicada II, Escola de Enxeñaría Aeronáutica e do Espazo, Campus As Lagoas, E-32004 Ourense, Spain
[4] Departamento de Física Aplicada, Facultade de Física, Universidade de Santiago de Compostela, E-15782 Santiago de Compostela, Spain

E-mail: xabier.prado@gmail.com, angel.paredes@uvigo.es, area@uvigo.gal, josemanuel.dominguez@usc.es and jorge.mira@usc.es

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Abstract

Throughout history, scales have served as instrumental tools for quantifying the weight of objects, relying on a comparative assessment against a specified reference weight. Scales featuring uneven arms, such as the bismar scale, have proven particularly adept at gauging masses within a specific range relative to a predetermined reference mass. On the other hand, the kinematics of elastic collisions hinge on the inertial masses of the colliding entities. By observing the aftermath of a collision between a known reference mass and an object of unknown mass, one can deduce the latter's mass. In this contribution, we highlight a fascinating and clear analogy between these two methodologies. We do so by adapting a geometric approach, initially applicable to the bismar scale, to both non-relativistic and relativistic elastic collisions, encompassing phenomena such as Compton scattering.

Keywords: education in physics, geometrical methods, classical mechanics, special relativity

* Author to whom any correspondence should be addressed.

1. Introduction

In the history of mankind there are a number of discoveries that have allowed great advances. Among them, we would like to emphasize being able to compare weights of objects which is essential for modern trade, activity that already began in the Neolithic period (about 7000–4000 b.C. depending on locations). The first evidence of scales and weights comes from Egypt and it dates back to circa 3000 b.C. [1]. It is possible to find representations of weighing scales in many ancient cultures such as China, from about 3rd to 4th century b.C. It can be interesting to notice that other systems of culture such as Aztecs seem not to have developed weighing systems [2].

The first idea for weighting is the most evident: equal-arm balances. We have to wait until about 500 b.C. to have a completely different idea: unequal-arm balances. The main difference is that, in the first case, the weight of the object to be weighted has to be on one arm and it has to be balanced by an identical weight on the other arm of the balance, with equal length. On the other hand, for the so-called unequal-arm balances, the equilibrium point is obtained by just adjusting the length of the arm, i.e. the compensation is no longer due to changes of weight but changes of length. Representations of this type of balances go back to a comedy of Aristophanes (421 b.C.). Nowadays the unequal-arm balances can be explained by using the law of lever, that is detailed in our work.

Unequal-arm balance is not a unique object, but a class that includes different instruments. The so-called bismar balance belongs to the class of unequal-arm balances, and it has a mobile fulcrum, which is the main difference as compared with other unequal-arm balances. It is also remarkable that the scale deviates from linearity. The etymology of this word points to an Slavo-Lithuanian word, despite the fact that this balance also existed in the ancient Pompei and therefore probably a Latin name might also exist, but we did not find traces of it.

All these balances gauge weight, which is proportional to the (passive) gravitational mass. On the contrary, inertial mass is a parameter that signifies a body's resistance to acceleration. Despite the equivalence principle asserting that inertial and gravitational mass are identical, they remain distinct in conceptual terms. Hence, it prompts the question of whether a system akin to a balance exists for determining inertial mass, involving a comparison between a reference mass and the mass under consideration. In this context, elastic collisions come to mind, drawing on our common experience that contact with a larger mass results in a greater momentum transfer. Remarkably, we will demonstrate a clear analogy between the bismar balance and the computation of mass, achieved by enforcing the conservation of momentum and energy in an elastic collision between two bodies of different masses. In particular, if the masses are equal, the scenario would mirror that of an equal-arm balance.

It has been asserted that analogies can enhance student learning under specific conditions, as discussed in [3] and related references. Accordingly, we anticipate that our discourse holds educational value, offering an alternative approach to exploring connections between fundamental concepts in mechanics, such as the law of the lever, mass, momentum transfer, and energy conservation.

Furthermore, our presentation relies on geometrical constructions. While both algebraic and geometric codifications of a given problem are equally valid, some students may prefer a geometrical approach over an analytical one, as highlighted in [4]. Even though the algebraic account may appear more general and formally simpler, certain mathematicians reject such simplicity, viewing it as lacking in the capacity to stimulate a deep understanding of the subject matter among students, as noted by [5]. Beyond its application in lower undergraduate education, we hope that our discussion might be of interest to graduate students and scholars, who may find it an interesting curiosity. Our approach offers a novel viewpoint on well-

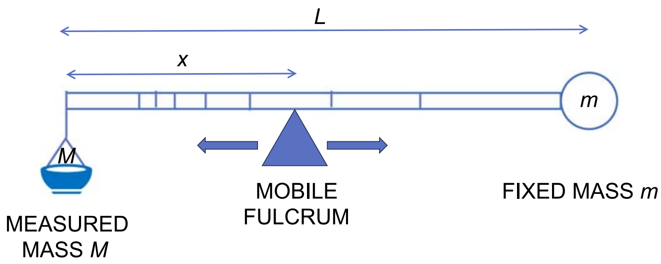

Figure 1. A schematic view of a bismar scale. For a mind-blowing picture of an artisanal bismar scale found in the ruins of Pompei, see [9].

known facts, with the hope that it proves stimulating and adds a new dimension to the understanding of the subject.

Our main purpose is to provide a geometrical construction for the determination of mass in a bismar scale and, then, to build analogous approaches for elastic collisions. In section 2 we introduce the notion of bismar scale, we indicate how to relate it to Archimedes' law of the lever and we depict a ruler-and-compass method to compute the mass from the position of the fulcrum at equilibrium. In section 3, we explain how the same ruler-and-compass method applies to the outcome of a one-dimensional elastic collision. In section 4, we generalize the construction to relativistic kinematics, including the limiting case of a massless particle. In section 5, we discuss collisions for which ingoing and outgoing momenta are not collinear, specifically illustrating the case of Compton scattering. Finally, in section 6 we summarize and conclude.

As supporting material for this contribution, we have created a website [6] featuring a collection of interactive apps that correspond to the various cases illustrated in the figures. These applets allow readers to manipulate parameters and observe the corresponding changes in results. Such virtual visual tools are particularly valuable for students as they augment the learning process, fostering engagement and motivation, as highlighted in references [7, 8].

2. A geometrical view of the bismar scale measurements

The bismar scale is a type of unequal arms scale that was developed in ancient times for the measurements of weight. It consists of a reference mass m that acts as a counterweight attached to the edge of a bar. At the other end of the bar, we place the mass to be measured, M. The fulcrum can be displaced in order to achieve an equilibrium.[5] This is schematically depicted in figure 1.

Since L and m are fixed for a given device, the mass M can be determined once x is known. Using Archimedes' law of the lever:

$$\frac{M}{m} = \frac{L - x}{x} \tag{1}$$

As an aside, it is worth mentioning that bismar scales were used long before the law of the lever was known, and the relation $M(x)$ was determined empirically [9]. This is an example in which technological developments helped in deriving scientific knowledge. In fact, the

[5] The bismar scale should not be confused with the Roman steelyard, perhaps the better known class of unequal arms scale. In a Roman steelyard, the fulcrum is fixed and the counterweight can be displaced. The geometrical methods described in this contribution could also be applied to a Roman steelyard, but they fit better the bismar scale.

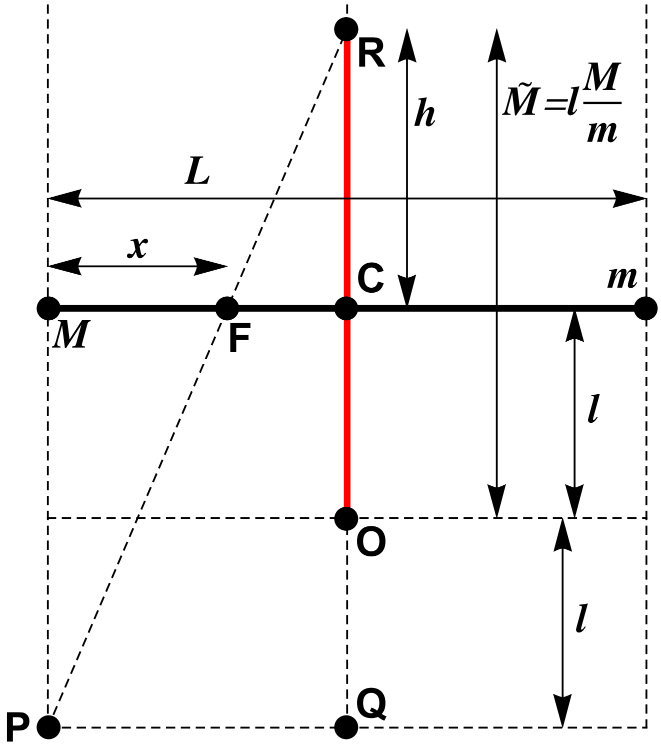

Figure 2. A geometric construction to find M from a bismar scale with counterweight m and fulcrum position in equilibrium at F.

influential *Mechanical Problems* by Aristotle, the oldest known treatise on Mechanics, were a discussion on how to abstract concepts such as force, load, etc from an analysis of simple technology [10].

Our goal now is to develop a geometrical method to compute M given an equilibrium as that depicted in figure 1, by adapting known constructions [9, 11]. We do so by constructing triangles in a way in which equation (1) can be related to a Thales theorem. This is shown in figure 2. The horizontal solid line represents the scale with M and m at the endpoints, with its center at point C and the fulcrum position for a given equilibrium is depicted by F. We then introduce an arbitrary fiducial length l and draw parallel lines at distances l and $2l$ from the scale, defining the points O, P and Q as shown in the figure. Extending the segment that joins P to F, we find the point R.

From the similar triangles PQR and FCR, we find:

$$\frac{L/2}{h+2l} = \frac{L/2 - x}{h} \qquad \Rightarrow \qquad h = l\frac{L-2x}{x} \tag{2}$$

Comparing to equation (1), we see that:

$$\overline{\mathrm{OR}} = h + l = l\frac{L-x}{x} = l\frac{M}{m} = \tilde{M} \tag{3}$$

Thus, by measuring the length of the OR segment, we can determine the mass M in terms of the reference quantities l, m. Notice that for the figure we have taken $M > m$ but the construction is equally valid for $M < m$. In the limiting case of $M \to 0$, the fulcrum would approach the right edge of the bar, which is aligned with P and O.

Even if the length-mass relation (1) could be implemented within simpler graphical constructions, the value of figure 2 is that it reproduces the actual geometry of the bismar scale. It solves the measuring of an unknown mass with a bismar scale as the one in figure 1, and it therefore encodes geometrically the fact that the point of equilibrium (the fulcrum) lies exactly where the total torque of external forces vanishes.

3. Nonrelativistic elastic collisions

The central idea of the present contribution is to put forward a formal analogy between the geometrical understanding of the bismar scale and that of elastic collisions. Other geometrical constructions for the illustration of elastic collisions in Newtonian mechanics have been presented in [12–14].

Consider the task of determining the mass, denoted as M, of a body that is initially at rest. This can be achieved by propelling a reference mass, represented as m, towards the target with an initial velocity denoted as v_i, which we can take to be positive without loss of generality. After the collision, it will have a different velocity v_f, which we assume to be collinear to the initial one (in section 6 we comment on the most general case). By measuring v_f, we can derive the value of the mass M. Notice the analogy with an unequal arms scale. The analog of the equal arms case would amount to looking for a reference mass m such that $v_f = 0$, meaning that $M = m$. Obviously, with this 'elastic collision scale', we are determining the inertial mass rather than the gravitational mass.

Denoting v_M the final velocity of the body of mass M, conservation of momentum and energy yield:

$$m\, v_i = m\, v_f + M\, v_M, \qquad \frac{1}{2}m\, v_i^2 = \frac{1}{2}m\, v_f^2 + \frac{1}{2}M\, v_M^2 \tag{4}$$

from which it is straightforward to prove:

$$\frac{M}{m} = \frac{v_i - v_f}{v_i + v_f} \tag{5}$$

which can be related to equation (1) if we identify $L \leftrightarrow 2v_i$, $x \leftrightarrow v_i + v_f$. With this identification, we can draw a geometrical construction for M by adapting the plot of figure 2. This is done in a $v - M$ diagram, for which it is natural to take m to play the role of the reference length l. The result is depicted in figure 3.

Once v_f is known, the point R is found by tracing the PF line and the value of M is given by the OR segment. In the example of the figure, v_f is negative and, as expected, $v_f \to -v_i$ would correspond to $M \to \infty$. On the other hand, positive v_f corresponds to $M < m$ and $M \to 0$ when $v_f \to v_i$.

In figure 2, the point F, which is pivotal in the geometrical construction, stands for *fulcrum*. On the other hand, in figure 3, the nomenclature F is also appropriate since it stands for the *final* velocity state of the reference mass m. Specifically, this parameter denotes the measurable quantity essential for the inference of the variable M.

Notice that in usual undergraduate education, the typical problems on elastic collisions consist in finding the final velocities given the initial velocities and the masses (for related geometrical constructions, see [12–14]). Our discussion here provides a complementary vision of the same kind of kinematics, in which the mass is inferred from the velocity. This is done to trace the analogy to weighing scales. However, it would be possible to set up this kind

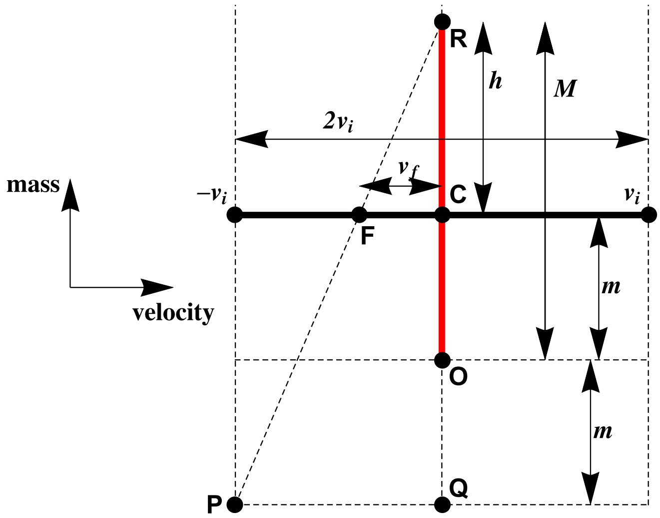

Figure 3. A geometric construction to find M from the outcome of an elastic non-relativistic collision. Notice the close similarity with the bismar scale of figure 2.

of situation in an undergraduate laboratory by working with sliders on a pneumatic bench. The implementation and analysis of such educational experiment is beyond the scope of the present work.

The geometrical representation of figure 3 highlights the analogy with section 2 and paves the way for understanding the following relativistic cases, which are much more demanding. Figure 3 differs from figure 2 in the magnitudes (velocities instead of arm lengths), and their striking geometric equivalence makes clear the basic fact that the fulcrum, in this case, corresponds to a reference system where both colliding objects have opposite momenta. This is exactly what the conservation laws of energy and momentum require for a collision to be elastic.

4. Relativistic elastic collisions

Special relativity is a more than a century-old theory that remains central to modern physics. However, some of its counter-intuitive features still make it challenging for developing efficient teaching and learning processes for secondary school students and undergraduates [15, 16]. Improving education has been one of the goals of developing geometrical and illustrative approaches to special relativity, see e.g. [17–20], including geometrical approaches to elastic collisions [13, 21, 22]. In this context, and building on the idea of using scales together with spacetime diagrams to delve into relativistic mass and energy [23], it is natural to wonder whether the analogy between figures 2 and 3 can be generalized to the relativistic case. We show below that there is indeed a simple generalization of figure 3 that does the job, by utilizing Minkowski diagrams in momentum space [24].

For a one-dimensional elastic collision in special relativity, conservation of momentum and energy read:

$$m\,\sigma_i = m\,\sigma_f + M\,\sigma_M, \qquad M + m\,\gamma_i = M\,\gamma_M + m\,\gamma_f, \tag{6}$$

where we are considering a reference particle of invariant mass m and initial velocity v_i impacting on a body, which is initially at rest, whose invariant mass M we want to determine.

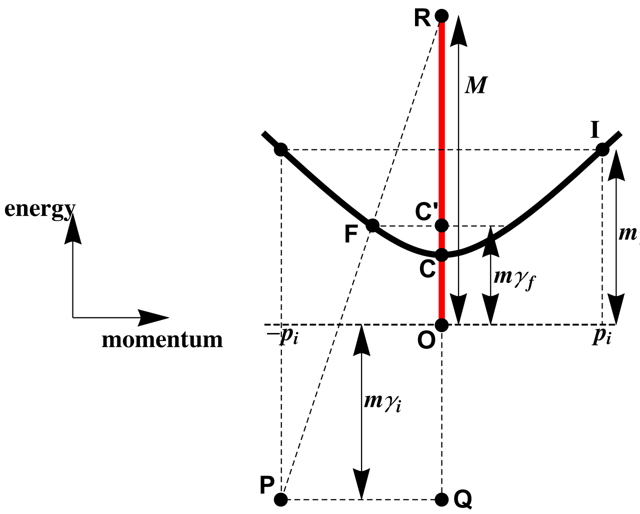

Figure 4. A geometric construction to find M from the outcome of an elastic relativistic solution.

After the collision, the particles have velocities v_f and v_M, respectively. For the different velocities, we have introduced the notation:

$$\gamma = \frac{1}{\sqrt{1-v^2}}, \qquad \sigma = \frac{v}{\sqrt{1-v^2}}, \tag{7}$$

where we use natural units $c = 1$. Notice that $\gamma \geqslant 1$ but σ can take any real value. Clearly, we have $\gamma^2 - \sigma^2 = 1$, which taking into account $E = m\gamma$, $p = m\sigma$, is proportional to the hyperbolic dispersion relation $E^2 - p^2 = m^2$. Equations (6), (7) lead to the relativistic generalization of equation (5), relating M to the initial and final velocities of the particle of mass m:

$$\frac{M}{m} = \frac{\gamma_f\sigma_i - \gamma_i\sigma_f}{\sigma_i + \sigma_f}. \tag{8}$$

We now outline the somewhat lengthy but straightforward computation to find (8) from (6) and the $\gamma^2 - \sigma^2 = 1$ identity that is immediately found from (7). The goal is to eliminate v_M from the equations. Making $\gamma_M^2 - \sigma_M^2 = 1$ from the expressions in (6), we get:

$$\frac{M}{m} = \frac{1 - \gamma_f\gamma_i + \sigma_i\sigma_f}{\gamma_f - \gamma_i}. \tag{9}$$

If we now multiply the fraction of the right-hand side by $\frac{\sigma_i + \sigma_f}{\gamma_f\sigma_i - \gamma_i\sigma_f}$ and we again use $\gamma^2 - \sigma^2 = 1$ for the initial and final velocities, we find 1. This therefore proves equation (8).

Taking advantage from (8), in figure 4 we show the geometric construction to find M graphically, in a very similar manner to the bismar scale. We use a Minkowski diagram in momentum space with the origin of coordinates at O and draw the hyperbola corresponding to $E^2 - p^2 = m^2$. We mark in the hyperbola the point corresponding to the initial (p_i, E_i) (point I) and final (p_f, E_f) (point F) states of motion (we assume $p_i > 0$). We also mark $(-p_i, E_i)$ and $(0, m)$ (point C). As in the bismar case, we draw two equispaced horizontal lines to find P. Finally, another horizontal from F leads to point C′. Again following figures 2 and 3 we

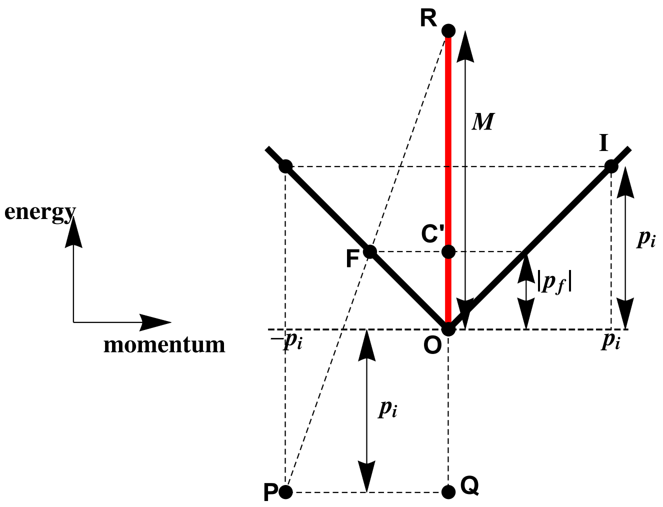

Figure 5. A geometric construction to find the mass of a mirror M from the energy loss of a reflected photon.

extend the PF segment to find R and the length of the OR segment is the sought mass M. This can be checked from the similarity of the triangles PQR and FC'R, that yields:

$$\frac{\overline{C'R}}{\overline{FC'}} = \frac{\overline{QR}}{\overline{PQ}} \quad \Rightarrow \quad \frac{M - m\gamma_f}{-m\sigma_f} = \frac{M + m\gamma_i}{m\sigma_i}. \tag{10}$$

This can be readily proved to be equivalent to equation (8). For given values of the reference mass m and its initial momentum p_i, all the points of the diagram are fixed except for the pivotal F which allows us to find R and thereby the mass M. The plot represents a case with $M > m$ that corresponds to $p_f < 0$. It is easy to see from the graph that $p_f > 0$ would yield $M < m$.

Let us now discuss the limiting case in which $m \to 0$ so the particle we are launching is massless. This situation can be described as a photon ($E = p$) that gets reflected from a mirror of mass M. The mirror acquires some velocity v_M and momentum $M\,\sigma_M$ and therefore the photon loses energy. We want to determine the mass of the mirror from the redshift of the photon. Conservation of energy and momentum are:

$$p_i = p_f + M\,\sigma_M, \qquad M + p_i = M\,\gamma_M + |p_f| \tag{11}$$

Taking $p_i > 0$, these equations have a solution with $p_f < 0$ and:

$$M = \frac{2p_i|p_f|}{p_i - |p_f|}. \tag{12}$$

We can construct an energy-momentum diagram for this case as the $m \to 0$ limit of figure 4. The result is depicted in figure 5. The similarity of the PQR and FC'R triangles yield:

$$\frac{M + p_i}{pi} = \frac{M - |p_f|}{|p_f|} \tag{13}$$

which corresponds to the relation given in equation (12).

5. Non-collinear Compton scattering

In the previous sections, we have addressed collinear collisions. It is only natural to inquire whether the construction can be extended to encompass the broader scenario wherein the direction of the final velocity remains arbitrary. That is in fact the case, and we can build on the one-dimensional results to address the more general cases. This concept embodies the essence of dimensional scaffolding, advocating for the examination of intricate physical relationships in lower dimensions. This approach facilitates a comprehensive understanding of their fundamental aspects during early learning phases, while ensuring that subsequent studies can seamlessly explore the complete dimensional complexity without hindrance [25].

The two-dimensional case can be solved with the one-dimensional method by finding an appropriate pivotal point by projection. On the other hand, the most general three dimensional case can always be reduced to the two-dimensional one by an appropriate rotation since the ingoing and outgoing velocity vectors always lie on a plane. In particular, we will demonstrate the adaptation of the scenario depicted in figure 5 to situations involving non-collinear momenta. Analogous constructions for figures 3 and 4 can be developed using a similar approach, though the details will not be provided in this discussion.

Let us consider a massless particle (say, a photon) with initial momentum along the x axis, and therefore with four-momentum $(p_i, p_i, 0, 0)$ (point I), where we assume $p_i > 0$. It impacts with a particle of mass M, initially at rest. After the elastic collision, the four-momentum of the outgoing photon is $(|p_f|, p_{f,x}, p_{f,y}, 0)$ (point F) with $|p_f| = \sqrt{p_{f,x}^2 + p_{f,y}^2}$ and that of the particle is $M(\gamma_M, \sigma_{M,x}, \sigma_{M,y}, 0)$. Clearly, this corresponds to the kinematics of Compton scattering. Preservation of energy and momentum leads to the following equations:

$$p_i + M = |p_f| + M\,\gamma_M, \quad p_i = p_{f,x} + M\sigma_{M,x}, \quad 0 = p_{f,y} + M\sigma_{M,y} \tag{14}$$

As in previous sections, we are interested in obtaining the mass M in terms of the photon momentum. From equation (14), we find:

$$M = p_i \frac{|p_f| - p_{f,x}}{p_i - |p_f|} \tag{15}$$

which, in the collinear case $|p_f| = -p_{f,x}$, clearly reduces to equation (12). We are interested in determining M from a geometrical construction in energy-momentum space. This is shown in figure 6. First, we find F′ as the projection of F onto the $p_y = 0$ plane, namely F′ has coordinates $(|p_f|, p_{f,x}, 0, 0)$. Then, we find F″ as the point of the light-cone $(q, -q, 0, 0)$ that is found by extending the IF′ segment. Once we have F″, we proceed as in figure 5, see figure 6.

Comparing figure 5 to figure 6, it is clear that from the similarity of the PQR and FC'R, we have:

$$M = \frac{2p_i q}{p_i - q} \tag{16}$$

which is simply equation (12) with $|p_f|$ replaced by q. Finally, we need to compute q in terms of the other quantities. Since F″, F′ and I are aligned by construction, comparing the triangles F′SF″ and ITF″ of the figure, we have:

$$\frac{|p_f| - q}{q + p_{f,x}} = \frac{p_i - q}{p_i + q} \tag{17}$$

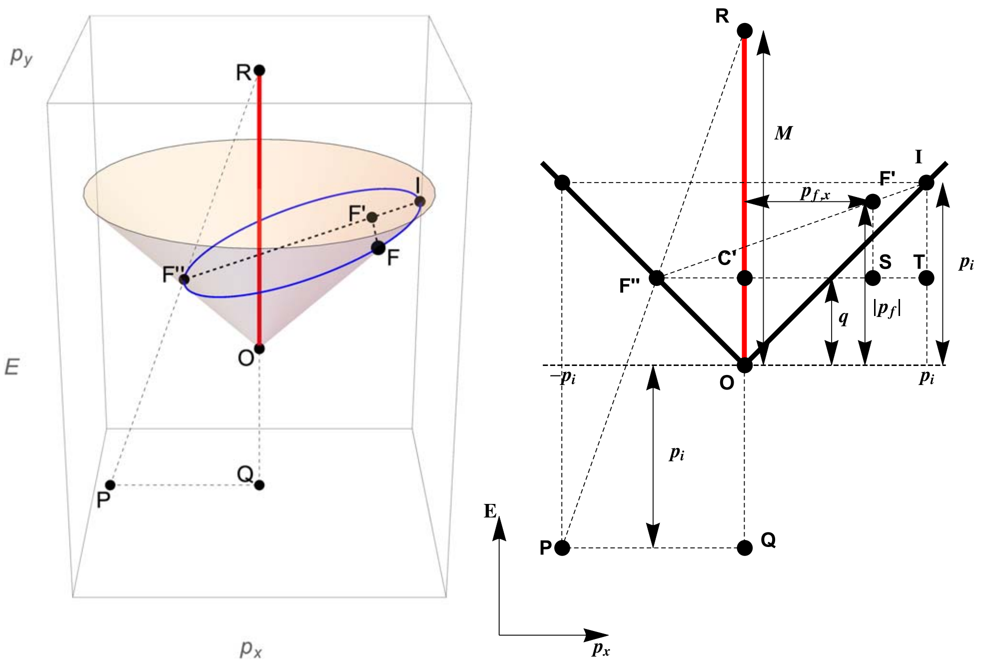

Figure 6. Finding M with the kinematics of Compton scattering, from the generalization of the geometrical method to determine masses in the bismar scale. On the left, we present a three dimensional plot that shows how to obtain the point F′ by projecting F and F″ by extending the IF′ segment. The blue ellipse is the intersection of the light cone with the plane that contains I, F and F′. Notice that any point of the ellipse for the final four-momentum gives rise to the same F″ and therefore it corresponds to a given mass M, as it should be expected since that is the mass of the particle. On the right, we show the $p_y = 0$ plane. Once the pivotal F″ point is known, the geometrical procedure to determine M follows as in figure 5.

from where we get:

$$q = p_i \frac{p_{f,x} - |p_f|}{p_{f,x} + |p_f| - 2p_i}. \tag{18}$$

Finally, inserting equation (18) in equation (16), we recover equation (15), completing the proof that the geometrical construction gives the correct result.

6. Summary and discussion

The weight of objects is closely linked to their gravitational mass and its measurement has been of great practical importance for millennia. Inertial mass is a different concept, measuring of an object's resistance to acceleration when a force is applied. Inertial masses are measured in different ways, but we have presented here a series of thought experiments from which it can be determined from the kinematic outcome of elastic collisions with some reference mass. Conceptually, this reference mass plays the same role as the known counterweight in a balance. We have shown that there is a clear formal analogy between these procedures, in which conservation of energy and momentum, together with their dispersion relation, take the role of Archimedes' law of the lever. In particular, we demonstrate the direct translation of a geometrical method used for determining mass equilibrium in a bismar scale, illustrated in figure 2, to the realm of nonrelativistic kinematics of collinear elastic collisions (figure 3). Expanding the applicability, we extend this methodology to relativistic collisions, leveraging Minkowski diagrams in momentum space (figure 4), encompassing phenomena like photon reflection depicted in figure 5. Finally, we showcase the generalization of this construction to non-collinear momentum transfer scenarios, exemplified by Compton scattering (figure 6).

It has been shown already [12–24] that geometric diagrams are helpful to relate relativistic problems to their non-relativistic counterparts, as well as to provide exact solutions. In particular [13], presents the concept of a spacetime lever, where the fulcrum embodies the reference frame in which the measurements are made simultaneously. If the lever is of the bismar type—that is, with a mobile fulcrum—the corresponding diagrams should show this fact visually. In the non-relativistic case, changing the reference frame does not affect the simultaneity, which remains always horizontal in spacetime, and this is the reason for the solid horizontal black line in figure 3. In the relativistic case, however, changing the reference frame alters the simultaneity, which is increasingly tilted following a hyperbolic line in spacetime due to the Lorentz transformation, as can be seen in figure 4. For massless particles like photons in Compton scattering, this hyperbola degenerates in two diagonal lines for the one-dimensional case, as in figure 5. In the most general case, the diagonal lines stretching in all directions build the so-called null cone represented in figure 6.

With these geometrical constructions, we aspire to offer a valuable resource for secondary school students and those in early undergraduate education, providing a means to solidify their grasp of fundamental concepts like relativistic and nonrelativistic kinematics, energy and momentum, as well as gravitational and inertial mass. Our approach involves an alternative methodology centered around analogies and geometrical constructions. Moreover, a set of interactive applets [6] should provide a valuable visual tool. Additionally, we hope to engage experienced readers with our presentation, offering a fresh perspective on familiar concepts. Our work endeavors to establish thought-provoking connections between disparate physical situations, illustratively expressed through the analogy of geometrical constructions for determining mass in diverse physical scenarios.

This teaching strategy implies a visual scaffolding through increasing dimensions, as explained in [25], and it has been already tested at the secondary school level, with interesting results [26]. The present paper offers an additional contribution in this direction.

Acknowledgments

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Data availability statement

No new data were created or analyzed in this study.

ORCID iDs

Xabier Prado https://orcid.org/0000-0001-9535-7499
Angel Paredes https://orcid.org/0000-0003-3207-1586
Jorge Mira https://orcid.org/0000-0002-6024-6294

References

[1] Büttner J 2013 Ancient balances at the nexus of innovation and knowledge *Research Topic des Max-Planck-Instituts für Wissenschaftsgeschichte* **32** Online publication https://www.mpiwg-berlin.mpg.de/news/features/features-feature32
[2] URL: https://es.wikisource.org/wiki/Página:Mitos_y_fantasías_de_los_aztecas.djvu/122 (visited on December 18th, 2023).
[3] Podolefsky N S and Finkelstein N D 2006 Use of analogy in learning physics: The role of representations *Phys. Rev. ST Phys. Educ. Res.* **2** 020101
[4] Galili I 2018 Physics and mathematics as interwoven disciplines in science education *Sci. Educ.* **27** 7
[5] Arnol'd V I 1998 On teaching mathematics *Russ. Math. Surv.* **53** 229
[6] Supplementary material: Interactive applets for the cases shown in figures 2–6 https://sites.google.com/view/spacetimebismar/home
[7] De Jong T, Linn M C and Zacharia Z C 2013 Physical and virtual laboratories in science and engineering education *Science* **340** 305
[8] Rau M A 2017 Conditions for the effectiveness of multiple visual representations in enhancing STEM learning *Educ. Psychol. Rev.* **29** 717
[9] Damerow P, Renn J and Rieger S 2002 Mechanical knowledge and Pompeian balances *Homo Faber: Studies on Nature, Technology, and Science at the Time of Pompeii* ed G Castagnetti and J Renn (L'Erma di Bretschneider) pp 93–108
[10] Renn J and McLaughlin P 2018 Emergence and expansion of preclassical mechanics. Boston studies in the philosophy and history of science *The Balance, the Lever and the Aristotelian Origins of Mechanics* ed R Feldhay, J Renn, M Schemmel and M Valleriani (Springer) vol 270
[11] Polhem (Polhaimer), C. 1716. Ett nytt och accurat sätt at uträkna Betzman. In: Daedalus hyperboreus, 41–50. Stockholm
[12] Ogura A 2017 Analyzing collisions in classical mechanics using mass-momentum diagrams *Eur. J. Phys.* **38** 055001
[13] Prado X, Area I, Paredes A, Domínguez-Castiñeiras J M, Edelstein J D and Mira J 2018 Archimedes meets Einstein: a millennial geometric bridge *Eur. J. Phys.* **39** 045802
[14] Ogura A 2018 Diagrammatic approach for investigating two dimensional elastic collisions in momentum space I: Newtonian mechanics *World J. Mech.* **8** 343
[15] Alstein P, Krijtenburg-Lewerissa K and van Joolingen W R 2021 Teaching and learning special relativity theory in secondary and lower undergraduate education: a literature review *Phys. Rev. Phys. Educ. Res.* **17** 023101
[16] Prado X, Domí nguez-Castiñeiras J M, Area I, Paredes A and Mira J 2020 Learning and teaching Einstein's theory of special relativity: state of the art arXiv: 2012.15149 (https://doi.org/10.25267/Rev_Eureka_ensen_divulg_cienc.2020.v17.i1.1103) Pubished (in Spanish) in *Revista Eureka sobre Enseñanza y Divulgación de las Ciencias 17 1103 and Revista de Enseñanza de la Física 32 107–122* (https://revistas.unc.edu.ar/index.php/revistaEF/article/view/28938)
[17] Mermin N D 1997 An introduction to space-time diagrams *Am. J. Phys.* **65** 476–86
[18] Takeuchi T 2010 *An Illustrated Guide to Relativity* (Cambridge University Press)
[19] Dray T 2017 The geometry of relativity *Am. J. Phys.* **85** 683–91
[20] Paredes A, Prado X and Mira J 2022 Relativistic velocity addition from the geometry of momentum space *Eur. J. Phys.* **43** 045601
[21] Bokor N 2011 Analysing collisions using Minkowski diagrams in momentum space *Eur. J. Phys.* **32** 773–82
[22] Ogura A 2018 Diagrammatic approach for investigating two dimensional elastic collisions in momentum space II: special relativity *World J. Mech.* **8** 353

[23] Prado X 2015 New light to relativity with levers and sticks hands-on science *Brightening our Future* ed M F Costa and B Vázquez-Dorrío (Hands-on Science Network) pp 49–60
[24] Saletan E J 1997 Minkowski diagrams in momentum space *Am. J. Phys.* **65** 799
[25] Prado X and Mira J 2021 Dimensional scaffolding of electromagnetism using geometric algebra *Eur. J. Phys.* **42** 015204
[26] Prado X 2010 A didactic proposal for the visual teaching of the theory of relativity in high school first course *Contemporary Science Education Research: Teaching* ed M F Tasar and G Cakmakci (Pegem Akademi) pp 297–305