What is a Congruence?

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Abstract.

1 Introduction

Lawvere's thesis contains a characterization of varieties of classical (finitary, one-sorted) algebras as categories with (1) coequalizers and kernel pairs, (2) an abstractly finite, regularly projective strong generator, and (3) effective congruences: every congruence (a reflexive, symmetric and transitive relation) is the kernel pair of some morphism. The concept of a congruence was generalized to the enriched setting in the fundamental paper [4], see also [5]. In case of enrichement over **Pos** (posets and monotone maps) or **Met** (metric spaces and non-expanding maps) we now present a much simpler concept called subcongruence for **Pos** and procongruence for **Met** (in order to distinguish them from the congruence in op. cit.).

2 Congruences in an order-enriched category

The role of a kernel pair of a morphism $f\colon X \to Y$ is played here by a *subkernel pair*: a universal pair $r, r'\colon R \to X$ with respect to $f \cdot r \leq f \cdot r'$. Every subkernel pair is a *relation on* X, i.e. the derived morphism $R \to X^2$ is an order-emedding, which is reflexive and transitive (and of course *not* symmetric). It is even *order-reflexive*: every parallel pair $u, u'\colon U \to X$ with $u \leq u'$ factorizes through r, r'.

A*subregular epimorphism* is a morphism which is the coinerter of a reflexive parallel pair.

Definition. A *subcongruence* on an object is an order-reflexive and transitive relation.

Let Σ be a classical (finitary) signature and denote by Σ-**Pos** the category of ordered algebras with monotone operations (and monotone homomorphisms). It has *effective subcongruences*: every subcongruence is the subkernel pair of some morphism.

More generally, every *variety of ordered algebras*, a full subcategory presented by a set of inequalities between terms, has effective subcongruences.

Here is the main result of [1] (improving that of [3]). An object is *subregularly projective* if its hom-functor preserves subregular epimorphisms.

Theorem. A **Pos**-enriched category is equivalent to a variety of ordered algebras iff it has (1) reflexive coequalizers and subkernel pairs, (2) an abstractly finite, subregularly projective strong generator, and (3) effective subcongruences.

3 Congruences in a metric-enriched category

For a real number $\varepsilon \geq 0$ the *ε-kernel pair* of a morphism $f: X \to Y$ is a universal pair $r_\varepsilon, r'_\varepsilon : R_\varepsilon \to X$ with respect to $d(f \cdot r_\varepsilon, f \cdot r'_\varepsilon) \leq \varepsilon$. Every such pair is a *relation*: the derived morphism to X^2 is an isometric embedding.

We introduce a weight $B: \mathcal{B}^{op} \to \mathbf{Met}$ such that every morphism f has a *kernel diagram* $D_f : \mathcal{B} \to \mathbf{Met}$ collecting all ε-kernel pairs of f. Every colimit of a diagram weighted by B is determined by a morphism. A *proregular epimorphism* is a morphism determining some colimit weighted by B. An object is *proregularly projective* if its hom-functor preserves proregular epimorphisms.

The kernel diagram D_f consists of parallel pairs that are reflexive and symmetric relations. They are also collectively transitive, and satisfy a continuity condition (expressing the fact that the map assigning R_ε to each $\varepsilon \geq 0$ preserves limits). A *procongruence* is a diagram weighted by B having all of those properties.

Mardare et al. [6] introduced varieties (aka 1-basic varieties) of quantitative algebras: they are categories of metric-enriched algebras presented by ε-equations between terms. We prove in [2] that up to equivalence they are precisely the **Met**-enriched categories which have (1) reflexive coequalizers and ε-coinserters, (2) an abstractly finite, proregularly projective strong generator, and (3) effective procongruences: every procongruence is the kernel diagram of some morphism.

References

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