Eilenberg-Moore categories and quiver representations of monads and comonads

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Abstract. Let Z be a scheme. Then, a famous result of Gabber (see, for instance, [4, Tag 077P]) shows that the category $QCoh(Z)$ of quasi-coherent sheaves over Z is a Grothendieck category. If S is a scheme and $\mathcal{Z}$ is an algebraic stack over S, the category $QCoh(\mathcal{Z})$ of quasi-coherent sheaves over $\mathcal{Z}$ is also a Grothendieck category (see, for instance, [4, Tag 06WU]).

In this paper, we prove a Gabber type result for representations of quivers in Eilenberg-Moore categories of monads. We develop a categorical framework for studying module representations taking values in Eilenberg-Moore categories of monads. For this, we generalize the usual setup of sheaves in several different ways. First, we replace the system of affine open subsets of a scheme by a quiver $\mathbb{Q} = (\mathbb{V}, \mathbb{E})$, i.e., a directed graph $\mathbb{Q}$ with a set of vertices $\mathbb{V}$ and a set of edges $\mathbb{E}$. This is motivated by Estrada and Virili [3] who studied modules over a representation $\mathcal{A} : \mathcal{X} \longrightarrow Add$ of a small category $\mathcal{X}$ taking values in small preadditive categories. Thereafter, we replace rings by monads over a given Grothendieck category $\mathcal{C}$. As such, we consider a representation $\mathscr{U} : \mathbb{Q} \longrightarrow Mnd(\mathcal{C})$ of the quiver $\mathbb{Q}$ taking values in the category $Mnd(\mathcal{C})$ of monads over $\mathcal{C}$. Finally, we replace the usual module categories over rings by Eilenberg-Moore categories of the monads over $\mathcal{C}$. By using systems of adjoint functors between Eilenberg-Moore categories, we obtain a categorical framework of modules over monad quivers. Our main objective is to give conditions for the category of modules over monad quivers to be Grothendieck categories, which play the role of noncommutative spaces.

We refer to a representation $\mathscr{U} : \mathbb{Q} \longrightarrow Mnd(\mathcal{C})$ as a monad quiver. To study modules over $\mathscr{U}$, we combine techniques on monads and adapt methods from earlier work in [1], [2] which are inspired by the cardinality arguments of Estrada and Virili [3]. One of our key steps is finding a modulus like bound for an endofunctor $U : \mathcal{C} \longrightarrow \mathcal{C}$ in terms of $\kappa(G)$, where G is a generator for $\mathcal{C}$ and $\kappa(G)$ is a cardinal such that G is $\kappa(G)$-presentable. As with usual ringed spaces, we have to study two kinds of

module categories over a monad quiver. The first behaves like a sheaf of modules over a ringed space. The second consists of modules that are cartesian, which resemble quasi-coherent sheaves. Another feature of our paper is that we study modules over a monad quiver in two different orientations, which we refer to as “cis-modules” and “trans-modules.” We establish similar results for comodules over a comonad quiver $\mathscr{V} : \mathbb{Q} \longrightarrow Cmd(\mathcal{C})$ taking values in comonads over $\mathcal{C}$. We conclude with rational pairings of a monad quiver with a comonad quiver, which relate comodules over a comonad quiver to coreflective subcategories of modules over monad quivers.

References

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