The pullback theorem for (relative) monads

N. Arkor

Nathanael Arkor (n@arkor.co)
Tallinn University of Technology

Dylan McDermott (dylan@dylanm.org)
Reykjavik University

Abstract.

The pullback theorem for monads states that, given a monad T on a category E, the category of T-algebras may be constructed by a certain pullback involving the category of free T-algebras, as given below [1].

$$\begin{array}{ccc} \mathbf{Alg}(T) & \hookrightarrow & [\mathbf{Kl}(T)^{\mathrm{op}}, \mathbf{Set}] \\ {\scriptstyle u_T}\downarrow & \lrcorner & \downarrow{\scriptstyle [k_T^{\mathrm{op}}, \mathbf{Set}]} \\ E & \underset{y_E}{\hookrightarrow} & [E^{\mathrm{op}}, \mathbf{Set}] \end{array}$$

Here, $u_T \colon \mathbf{Alg}(T) \to E$ is the forgetful functor, $y_E \colon E \to [E^{\mathrm{op}}, \mathbf{Set}]$ is the Yoneda embedding, and $k_T \colon E \to \mathbf{Kl}(T)$ is the Kleisli inclusion.

As has been repeatedly demonstrated in recent years, many classes of well-behaved monads admit a refinement of the pullback theorem, in which the category of algebras is expressed as a pullback over the nerve $n_j \colon E \to [A^{\mathrm{op}}, \mathbf{Set}]$ of some dense functor $j \colon A \to E$. (In particular, the classical pullback theorem is recovered by taking $j = 1_E$.) Such characterisations are frequently referred to as *nerve theorems*, and endow the monad and its category of algebras with particularly nice properties. Examples of classes of monads satisfying respective nerve theorems include *familially representable monads* [2], *monads with arities* [3, 4, 5], *$\mathscr{J}$-ary monads* [6], and *nervous monads* [7, 8]. It is natural to wonder to what extent these phenomena are related.

In this talk, I will explain how, by generalising the pullback theorem from monads to *relative monads* [9, 10], we may view each of the aforementioned nerve theorems as particular instances of a more (and, indeed, maximally) general phenomenon.

This talk is based on the recent preprint [11].

References

[1] F. E. J. Linton, *An outline of functorial semantics*, Seminar on triples and categorical homology theory, Springer, 1969.

[2] T. Leinster, *Nerves of algebras*, Talk presented at Category Theory 2004.

[3] M. Weber, *Familial 2-functors and parametric right adjoints*, Theory and Applications of Categories, 2007.

[4] P.-A. Melliès, *Segal condition meets computational effects*, 25th Annual IEEE Symposium on Logic in Computer Science, 2010.

[5] C. Berger, P.-A. Melliès, and M. Weber, *Monads with arities and their associated theories*, Journal of Pure and Applied Algebra, 2012.

[6] R. Lucyshyn-Wright, *Enriched algebraic theories and monads for a system of arities*, Theory and Applications of Categories, 2016.

[7] J. Bourke, and R. Garner, *Monads and theories*, Advances in Mathematics, 2019.

[8] R. Lucyshyn-Wright, and J. Parker, *Enriched structure-semantics adjunctions and monad-theory equivalences for subcategories of arities*, arXiv preprint.

[9] T. Altenkirch, J. Chapman, and T. Uustalu, *Monads need not be endofunctors*, International Conference on Foundations of Software Science and Computational Structures, 2010.

[10] N. Arkor, and D. McDermott, *The formal theory of relative monads*, Journal of Pure and Applied Algebra, 2024.

[11] N. Arkor, and D. McDermott, *The pullback theorem for relative monads*, arXiv preprint.