Algebraic Type Theory

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Abstract.

A type theoretic universe $\pi : E \to U$ in a locally cartesian closed category $\mathcal{C}$ (as in [1]) can be shown to bear an algebraic structure resulting from the type-forming operations of unit type, dependent sum, and dependent product (as shown in [2]). Specifically, the associated polynomial endofunctor $P_\pi : \mathcal{C} \to \mathcal{C}$ has the structure of a monad, for which π is itself an algebra.

This structure is here abstracted to form the concept of a "Martin-Löf algebra". Any ML-algebra is shown to model Martin-Löf type theory, and the free ones then have special type-theoretic properties. The general theory of ML-algebras is a "proof-relevant" or *categorified* version of the theory of Zermelo-Fraenkel algebras from the algebraic set theory of Joyal & Moerdijk [3].

For example, any representable natural transformation $\pi : E \to U$ of presheaves, as in [2], is necessarily *tiny* in the sense of Lawvere: the right adjoint push-forward functor $\pi_* : \mathcal{C}/E \to \mathcal{C}/U$ has a further right adjoint. It follows that the polynomial endofunctor $P_\pi : \mathcal{C} \to \mathcal{C}$ is cocontinuous and therefore admits an algebraically free monad structure, by a familiar iteration [4]. The (type theory modeled by the) colimit $\pi^\omega : E^\omega \to U^\omega$ is then the free completion under Σ-types of (that modeled by) $\pi : E \to U$. Various other type-theoretic constructions are similarly related to functorial algebraic ones.

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References

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