2-stacks over bisites

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Abstract.

Stacks generalize one dimension higher the fundamental concept of sheaf. They are pseudofunctors that are able to glue together weakly compatible local data into global data. Stacks are a very important concept in geometry, due to their ability to take into account automorphisms of objects. While many classification problems do not have a moduli space as solution because of the presence of automorphisms, it is often nonetheless possible to construct a moduli stack.

In recent years, the research community has begun generalizing the notion of stack one dimension higher. Lurie studied a notion of $(\infty, 1)$-stack, that yields a notion of $(2, 1)$-stack for a trihomomorphism that takes values in $(2, 1)$-categories, when truncated to dimension 3. And Campbell introduced a notion of 2-stack that involves a trihomomorphism from a one-dimensional category into the tricategory of bicategories.

In this talk, we will introduce a notion of 2-stack that is suitable for a trihomomorphism from a 2-category endowed with a bitopology into the tricategory of bicategories. The notion of bitopology that we consider is the one introduced by Street in [4] for bicategories. We achieve our definition of 2-stack by generalizing a characterization of stack due to Street [4].

Since our definition of 2-stack is quite abstract, we will also present a useful characterization in terms of explicit gluing conditions that can be checked more easily in practice. These explicit conditions generalize to one dimension higher the usual stacky gluing conditions. A key idea behind our characterization is to use the tricategorical Yoneda Lemma to translate the biequivalences required by the definition of 2-stack into effectiveness conditions of appropriate data of descent. As a biequivalence is equivalently a pseudofunctor which is surjective on equivalence classes of objects, essentially surjective on morphisms and fully faithful on 2-cells, we obtain effectiveness conditions for data of descent on objects, morphisms and 2-cells. It would have been hard to give the definition of 2-stack in these explicit terms from the beginning, as we would not have known the correct coherences to ask in the various gluing conditions. Our natural implicit definition is instead able to guide us in finding the right coherence conditions. Our definition of 2-stack and our characterization in terms of explicit gluing conditions have been developed in [2].

Finally, we will present the motivating example for our notion of 2-stack, which is the one of quotient 2-stack. In [1], we generalized principal bundles and quotient stacks to the categorical context of sites. We then aimed at a generalization of our theory one dimension higher, to the context of bisites, motivated by promising applications of principal 2-bundles to higher gauge theory. But there was no notion of higher dimensional stack suitable for the produced analogues of quotient prestacks in the two-categorical context. Our notion of 2-stack is able to fill this gap. Indeed, we have proven that, if the bisite satisfies some mild conditions, our analogues of quotient stacks one dimension higher are 2-stacks. Our theory of principal 2-bundles and quotient 2-stacks has been developed in [3].

Quotient 2-stacks could give a substantial contribution towards the development of a cohomology theory of schemes, and more in general of stacks, with coefficients in stacks of abelian 2-groups. This theory would produce new and refined 2-categorical invariants associated to schemes and algebraic stacks, that could solve numerous open problems in algebraic geometry.

References

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