Crossed modules of algebras over an operad and an application to rational homotopy theory

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Abstract.

Crossed modules have been studied in various contexts for a long time in algebraic topology, beginning with the work of Whitehead in [1] to understand pointed homotopy 2 types. Loday in [2] shows that crossed modules of groups can be understood in many ways : as groups internal to the category of small categories, as simplicial groups with Moore complex of length 1, as a $1-$cat-groups, or as a categories internal to the category of groups. These last two descriptions allow him to generalize crossed modules of groups to higher versions, namely n-cat-groups. Morally such an object is an n-fold category internal to groups, and higher versions of crossed module, say n-crossed modules can be inductively defined as a crossed module internal to the category of $(n-1)$-crossed modules. The other idea of Loday is that one can associate to an $n-$crossed module $\mathcal{L} = \mathbb{H} \overset{d}{\to} \mathbb{G}$ a space $B\mathcal{L}$ such that the canonical sequence $B\mathbb{H} \to B\mathbb{G} \to B\mathcal{L}$ is a homotopy fiber sequence. Using these ideas, he was able to prove that n-crossed modules are models for pointed homotopy $(n+1)$-types. Later on, several authors studied crossed modules in other algebraic contexts. For example Ellis in [3] studies n-crossed cubes of associative/commutative/Lie/etc.. algebras and proves that such objects can be equivalently defined as n-fold categories internal to algebras of the associated type. In the 00's, Janelidze in [4] gives a general framework in which crossed modules can be defined : semi-abelian categories, and proves that the category of crossed modules internal to a semi-abelian category $\mathcal{C}$ is equivalent to the category $Cat(\mathcal{C})$ of internal categories to $\mathcal{C}$. One can play the same game and prove that n-fold crossed modules internal to $\mathcal{C}$ are equivalent to n-fold categories internal to $\mathcal{C}$. Many algebraic categories are known to be semi-abelian : the categories of groups, non-unital rings, associative or commutative algebras, Lie algebras, etc ... And it seems to be folklore that the category of algebras over a symmetric algebraic reduced operad is semi-abelian.

However the definitions of Janelidze and Ellis are not so well adapted to the case of algebras over an operad. First of all, their definitions are a bit involved, Indeed they both required lots of axioms. For example, a crossed module of associative algebras is the data of a morphism of algebras $d : A \to B$, an internal action of B on A such that d is equivariant and satisfies a "Peiffer" condition. When one wants to go to higher crossed modules, this becomes awful. Second, the homotopical properties, and especially the link between (higher) crossed modules of algebras and the homotopy theory of algebras over an operad is not clear at all.

In a current work, Leray-Rivière-Wagemann (LRW) give yet another definition of crossed modules of algebras over an operad $\mathcal{P}$, namely it is the data of a $\mathcal{P}$-algebra structure on a chain complex $...0 \to A \overset{d}{\to} B$ concentrated in degrees 0 and 1. This has two main advantages over the previous

definitions. First it is a very concise and clear definition. Second it is clearly linked to the homotopy theory of $\mathcal{P}$-algebras, that is differential graded $\mathcal{P}-$algebras.

In an upcoming series of two papers, we generalize the definition of LRW to get a very concise definition of crossed n-cube of algebras over an operad, namely it is the data of a $\mathcal{P}-$algebra structure on an $n-$fold chain complex $A_{\bullet,...,\bullet}$ concentrated in degrees $(\epsilon_1, ..., \epsilon_n), \epsilon_i \in \{0, 1\}$. We prove that such a structure descends to $\mathcal{P}-$algebra structures on each $A_{\epsilon_1,...,\epsilon_n}$, and that we get a crossed n-cube of algebras in the sense of Ellis. We also prove that our category of crossed n-cubes of $\mathcal{P}-$algebras is equivalent to the category of $n-$fold categories internal to the category of $\mathcal{P}-$algebras, so our definition is equivalent to the ones of Ellis and Janelidze. The main advantage of our "global" definition as opposed to the previous "locals" ones lies in the existence of the monoidal functor $Tot^{\oplus} : Ch(...Ch(\mathcal{A})) \to Ch(\mathcal{A})$ as soon as $\mathcal{A}$ is a monoidal abelian category, for example $\mathcal{A} = Vect_{\mathbb{Q}}$. In particular it induces a functor which sends a crossed $n-$cube of $\mathcal{P}$-algebras (in our sense) to a differential graded $\mathcal{P}$-algebra concentrated in degrees $0, ..., n + 1$, so here the link with the homotopy theory of $\mathcal{P}$-algebras is almost for free.

The second paper of this series is devoted to an application to rational homotopy theory and especially to a conjecture of Félix and Tanré in [?]. In this paper they construct a crossed module of groups $C(\mathfrak{g})$ associated to a complete differential graded Lie algebra $\mathfrak{g}$ concentrated in degrees 0 and 1. They prove that the classifying space $BC(\mathfrak{g})$ of this crossed module is isomorphic to the geometric realization $\langle \mathfrak{g} \rangle$ of $\mathfrak{g}$ in the sense of [6] and conjecture that given a cdgl concentrated in degrees $0, ..., n$, one can associated to it a n-cat-group and that its classifying space is isomorphic to the geometric realization of this Lie algebra. We studied the case $n = 2$ and prove that for a Lie algebra in square $\mathfrak{g}$, one can associate $Tot(\mathfrak{g})$ a complete dgLie algebra of height 3 and its geometric realization $\langle Tot(\mathfrak{g}) \rangle$. Or one can associate a crossed square of groups and consider its classifying space. We prove that thoses two constructions are weakly homotopy equivalent.

References

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