Configuration spaces of points and degenerate higher categories

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Abstract.

Degenerate higher categories are those in which a certain number of the lowest dimensions are trivial, and these structures provide a good test case for theories of n-category. Via a dimension shift, a k-degenerate $(n+k)$-category can be regarded as an n-category with extra structure, allowing us to study higher-dimensional structures via lower-dimensional ones that might be better understood.

Thus far the theory of these structures is not very well-developed, but another issue is the paucity of examples. In this talk we will present a series of examples derived from the fundamental n-groupoids of $\mathbf{2}$, the indiscrete space with two points; here $n \leq \infty$. This may seem simple-minded (among other reasons because the space is contractible) but it provides a starting point for several interesting insights into the issues of degeneracy and commutativity.

One of the main ideas of the topic is that if the lowest k dimensions of a higher category are trivial we can disregard them, and consider the k-cells to be the 0-cells of a new, lower-dimensional structure. The k types of composition they had as k-cells become k monoidal structures on the new lower-dimensional structure, which results in the following slogan:

A k-degenerate $(n+k)$-category "is" a k-tuply monoidal n-category.

Another key idea is that the k monoidal structures can be seen to result in certain types of commutativity, via generalised Eckmann–Hilton arguments. The structures that are expected to arise are organised into the "periodic table of n-categories" conjectured in [1]. For example, a 2-degenerate 3-category "is" a 2-tuply monoidal category, which is in turn seen, via a weak Eckmann–Hilton argument, to be a braided monoidal category.

The general theory of these structures is not very rigorously understood, even in terms of the basic definitions, let alone how we perform the dimension shift and convert the compositions into monoidal structures, how the generalised Eckmann–Hilton arguments work, what structures arise from those arguments, and what "is" really means.

In this talk we work with Trimble's definition of higher category, in which composition is parametrised by operad actions. However, we will crucially use the little intervals operad $\mathcal{C}_1$ rather than the universal operad acting on path spaces as originally specified by Trimble. The definition comes with an

immediate notion of fundamental n-groupoid; here n can also be ∞, by means of the work of [2]. Our first suite of examples then is the fundamental n-groupoid of $\mathbf{2}$, for $n \leq \infty$. The k-cells are essentially continuous maps $I^k \to \mathbf{2}$, and as $\mathbf{2}$ is the subobject classifier of Top we can regard these as subspaces of k-cubes. (Here I denotes the closed interval $[0,1]$.) Composition proceeds by "stacking" the cubes in any of the k possible directions, and reparametrising via the little intervals operad.

We can then restrict our attention to the k-degenerate version, where for all $j \leq k$ the only j-cell is the empty subspace of I^j. This produces a (relatively) concrete example of k-degenerate $(n+k)$-categories, and we show how to regard this as a k-tuply monoidal n-category.

Our more specific example of interest comes from restricting our attention further, to study configurations of points in n-space. We construct an n-degenerate $(n+1)$-category derived from the fundamental ∞-groupoid of $\mathbf{2}$ as follows:

- For $j < n$ there is just one j-cell, the empty subspace of I^j.
- The n-cells are the *finite* subspaces of I^n, where all lower-dimensional boundary cells are the empty subspace; thus these amount to configurations of points in the interior of I^n.
- The $(n+1)$-cells are braids, realised as subspaces of I^{n+1}.

The properties of the little intervals operad $\mathcal{C}_1$ enable us to prove that these cells are closed under composition parametrised by $\mathcal{C}_1$. This gives us, for all $n \geq 1$, an n-fold monoidal category of configurations of points in I^n and braids between them.

This will enable us, in a sequel, to study n-degenerate $(n+1)$-categories in generality, and, following our work in [3, 4], exhibit a biequivalence between a suitable 2-category of n-degenerate $(n+1)$-categories and the 2-category of symmetric monoidal categories.

References

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