The smothering model structure on **Cat**

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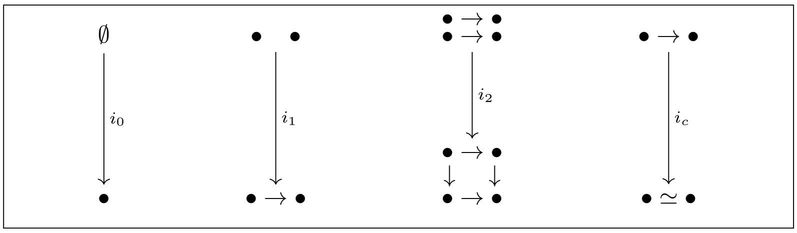
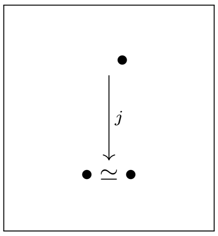
Abstract.

The notion of smothering functor introduced by Riehl and Verity ([1]) defines a class of functors which extends the usual notion of equivalence of categories by relaxing the faithfulness condition. These functors appear naturally as comparison functors between homotopy categories, for instance the comparison functor relating the homotopy category of a pullback of quasicategories to the pullback of the homotopy categories of these quasicategories.

In this work, we consider a slight strengthening of the original definition of smothering functors, which we refer to as stably smothering, that actually form the trivial fibrations of a right Bousfield localization of the natural model structure on **Cat**. Precisely, the generating cofibration are given by the inclusion i_k (for $k \geq 0$) of two opposite face of the k-cube in the k-cube as well as i_c. Those are pictured for $0 \leq k \leq 2$ on the left below. The generating trivial cofibration is j, as on the right below





The weak equivalences for this model structure are the weakly stably smothering functors. They can be defined either as the functors which are essentially surjective on objects and lift against all the generating cofibration, except possibly i_0, or equivalently as the functors having the 2-categorical right lifting property against the generating cofibration. Unlike weakly smothering functors in the sense of [1], they enjoy the 2-ouf-of-3 property, which is crucial to prove the following:

Theorem 1. *There is a cofibrantly generated model structure on* **Cat** *with weak equivalences the weakly stably smothering functors.*

In parallel, we introduce an equivalence relation, *indiscernibility*, on parallel isomorphisms in a given category C. The starting point is an attempt to quotient out all parallel isomorphisms as to discard the structure provided by a given isomorphism $\alpha : x \simeq y$ between two objects, and only keep the mere existence of such an isomorphism between x and y. It turns out that such an identification is not necessarily compatible with the categorical structure of C. To make this statement precise, we consider the simplicial object $\mathbf{core}(C^{\to}_{\bullet})$ in **Gpd** whose groupoid of n-simplices is that of paths of arrows in C of length n and isomorphisms between two such paths. Given a relation R on parallel isomorphism, there is a quotient $|\mathbf{core}(C^{\to}_{\bullet})|_R$ whose groupoid of n-simplices has arrows the equivalence class of isomorphisms between paths of length n in C (with respect to the pointwise equivalence relation on isomorphisms between two given paths deduced from R). While this simplicial object need not satisfy the Segal condition for an arbitrary relation, there exists an equivalence relation $\mathcal{R}$ on parallel isomorphisms which can be characterized as the maximal relation R such as the simplicial object $|\mathbf{core}(C^{\to}_{\bullet})|_R$ yields a category object in **Gpd**. One can then observe that weakly smothering functors preserve and reflect indiscernibility and admit the following characterization:

Proposition 2. *A functor $F : C \to D$ is weakly smothering if and only if it preserves indiscernibility and the induced transformation $|\mathbf{core}(C^{\to}_{\bullet})|_{\mathcal{R}_C} \to |\mathbf{core}(D^{\to}_{\bullet})|_{\mathcal{R}_D}$ is a pointwise equivalence of groupoids.*

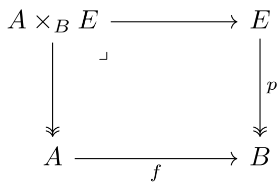
The indiscernibility relation $\mathcal{R}$ can be extended to a relation $\mathcal{R}'$ between parallel arrows which is a congruence with respect to composition of arrows, and the canonical quotient map $C \to \Pi(C)$ is a smothering functor, where $\Pi(C)$ is the category with the same object as C and with morphisms the indiscernibility classes of arrows modulo $\mathcal{R}'$. The weakly smothering functors are equivalently those inducing an equivalence of categories $\Pi(C) \to \Pi(D)$.

The following example, adapted from Proposition 3.3.14 of [2], is archetypical:

Example 3.
Given a pullback squares of quasicategories as on the right with p an isofibration, the canonical functor

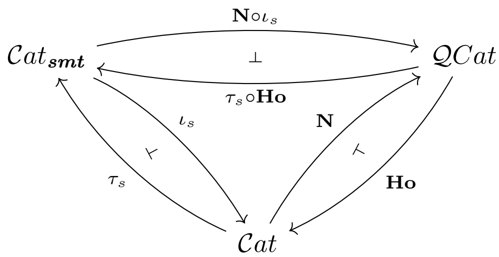
$$Ho(A \times_B E) \to Ho(A) \times_{Ho(B)} Ho(E)$$

is a weakly stably smothering functor.



This can in fact be account for the following important result, which states that, in the adjunction between the homotopy category functor **Ho** and the nerve functor **N**, the left and right adjoints are swapped when restricting to the ∞-coreflection of $\mathcal{C}at$ given by the smothering model structure:

Theorem 4. *There is a diagram of ∞-adjunctions*



where $\mathcal{C}at_{\boldsymbol{smt}}$, $\mathcal{C}at$ and $\mathcal{Q}\mathcal{C}at$ are the $(\infty, 1)$-categories presented by the smothering model structure, the natural model structure on **Cat** *and the Joyal model structure on simplicial sets respectively.*

References

[1] E. Riehl and D. Verity,*Elements of ∞-Category Theory*, Cambridge University Press, vol. 194, 2022.

[2] E. Riehl and D. Verity, *The 2-category theory of quasi-categories*, Advances in Mathematics 25 (2024), vol. 280, 549–642.