Differential bundles in Goodwillie calculus

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Abstract.

In recent work with Kristine Bauer and Matthew Burke [1] we developed the theory of *tangent ∞-categories*, the ∞-categorical version of Rosický's [9] and Cockett and Cruttwell's [3] theory of tangent categories. This theory allows us to make precise the analogy between Goodwillie's calculus of functors in homotopy theory [6] and the ordinary differential calculus of smooth manifolds, by constructing a tangent ∞-category whose objects are ∞-categories and whose morphisms are functors. The tangent bundle on an ∞-category $\mathcal{C}$ is that constructed by Lurie [7]: the ∞-category $T\mathcal{C}$ of *parameterized spectra* (in the sense of stable homotopy theory) over objects of $\mathcal{C}$.

Construction of the Goodwillie tangent structure opens the door for extending other ideas from tangent categories, and hence from smooth manifolds, to the functor calculus setting. In this talk, I will describe joint work with Kaya Arro in which we establish the analogues of smooth vector bundles. Cockett and Cruttwell [5] developed a notion of *differential bundle* in an arbitrary tangent category, and MacAdam showed in [8] that in the category of smooth manifolds that notion recovers precisely the smooth vector bundles (of locally constant rank). Our main result identifies a differential bundle in the Goodwillie tangent structure with a collection of stable ∞-categories and exact functors between them, parameterized by a base ∞-category.

Our work comprises three parts. First we extend Cockett and Cruttwell's definition of differential bundle to the setting of tangent ∞-categories. Then we show how any differential bundle can be recovered, up to equivalence, from its projection map and zero section by appropriate pullbacks. (This characterization is inspired by MacAdam's work but appears to be new, even for ordinary tangent categories.) Finally, we apply that construction to the Goodwillie tangent structure and establish our classification of differential bundles (and linear maps between them) in that setting.

Given the prominent role that vector bundles play in the theory of manifolds, we expect differential bundles to be central to the tangent category perspective on Goodwillie calculus. For example, we hope our framework will allow for a concrete definition of the *cotangent* bundle on an ∞-category. Concepts such as connections, torsion and curvature [4] and Lie algebroids [2] have also been defined in an arbitrary tangent category, and we expect these notions could now be identified in the world of ∞-categories too.

References

[1] K. Bauer, M. Burke, and M. Ching, *Tangent ∞-categories and Goodwillie calculus*, (2023), arxiv:2101.07819v4.

[2] M. Burke and B. MacAdam, *Involution algebroids: a generalisation of Lie algebroids for tangent categories*, 2019, arxiv:1904.06594.

[3] J. R. B. Cockett and G. S. H. Cruttwell, *Differential structure, tangent structure, and SDG*, Appl. Categ. Structures **22** (2014), no. 2, 331–417.

[4] J. R. B. Cockett and G. S. H. Cruttwell, *Connections in tangent categories*, Theory Appl. Categ. **32** (2017), Paper No. 26, 835–888.

[5] J. R. B. Cockett and G. S. H. Cruttwell, *Differential bundles and fibrations for tangent categories*, Cah. Topol. Géom. Différ. Catég. **59** (2018), no. 1, 10–92.

[6] T. G. Goodwillie, *Calculus. III. Taylor series*, Geom. Topol. **7** (2003), 645–711 (electronic).

[7] J. Lurie, *Higher Algebra*, (2017), available at `www.math.ias.edu/~lurie/`.

[8] B. MacAdam, *Vector bundles and differential bundles in the category of smooth manifolds*, Appl. Categ. Structures **29** (2021), no. 2, 285–310.

[9] J. Rosický, *Abstract tangent functors*, Diagrammes **12** (1984), JR1–JR11.