Fibered elementary quotient completion

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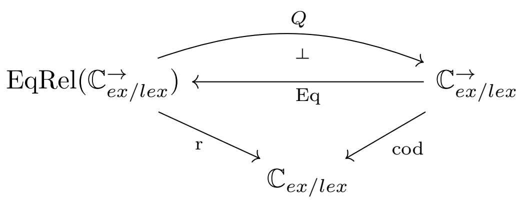
Abstract.

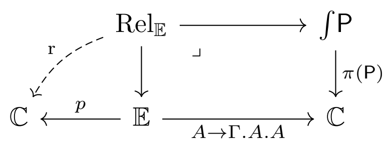
Completions of categories by quotients have been deeply studied in category theory. The main construction is the free exact completion of a (weakly) left exact category provided in [1, 2]. Later in [3], Maietti and Rosolini introduced the elementary quotient completion in order to give an abstract description of the *quotient model* in [4]. The main novelty is that the authors relativize the notions of equivalence relation and quotient for Lawvere's elementary doctrines, which are suitable functors of the form $\mathsf{P} : \mathbb{C}^{\mathrm{op}} \to \mathsf{Pos}$, from a category $\mathbb{C}$ with strict finite products to the category Pos of posets and order preserving functions. As shown in [3], the elementary quotient completion generalizes the exact completion of (weakly) left exact categories.

The present work originates from the following observation: the exact completion of a left exact category $\mathbb{C}$ not only adds stable quotients of equivalence relations; it also provides *fibered* quotients with respect to the codomain fibration, in the sense that there exists a fibered adjunction $Q \dashv \mathrm{Eq}$ as in the following diagram



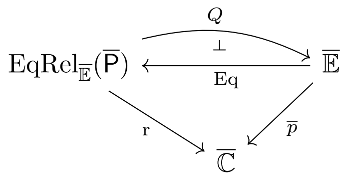
where the left hand fibration is that of equivalence relations on $\mathbb{C}^{\to}_{ex/lex}$, i.e. congruences $r_1, r_2 : z \to x$ in $\mathbb{C}^{\to}_{ex/lex}$ and $\mathrm{r}(r_1, r_2) := \mathrm{cod}(x)$, and Eq is the functor sending an object $x \in \mathbb{C}^{\to}_{ex/lex}$ to the pair (id_x, id_x). Pointwisely, the above diagram states that the slices of $\mathbb{C}_{ex/lex}$ have quotients of equivalence relations, which is a consequence of the fact the slices of $\mathbb{C}_{ex/lex}$ are also exact.

In this talk, we aim to generalize the above situation and to freely add fibered quotients with respect to an elementary doctrine $\mathsf{P} : \mathbb{C}^{\mathrm{op}} \to \mathsf{Pos}$ for a fibered category $p : \mathbb{E} \to \mathbb{C}$ with the same base category. To this extent, we assume that p is a comprehension category as in [5] in order to take into account P-equivalence relations on objects of $\mathbb{E}$ given by the subfibration on equivalence relations $\mathrm{r} : \mathrm{EqRel}_{\mathbb{E}}(\mathsf{P}) \to \mathbb{C}$ of the fibration of relations obtained through the following change of base situation



where $\Gamma.A$ is the common notation for the domain of the comprehension of an element $A \in \mathbb{E}$ over Γ, and where $\pi(\mathsf{P})$ is the Grothendieck construction applied to P.

Starting from a suitable pair of fibrations $(\pi(\mathsf{P}), p)$ where P is an elementary doctrine and p is a comprehension category, we provide a pair $(\pi(\overline{\mathsf{P}}), \overline{p})$ where $\overline{\mathsf{P}} : \overline{\mathbb{C}}^{\mathrm{op}} \to \mathsf{Pos}$ is the elementary quotient completion of P and $\overline{p}$ is a comprehension category with equality and fibered quotients as defined in [5], i.e. there exists a fibered adjunction as in the following diagram



$\overline{p}$ is obtained considering suitable *pseudo descent objects*, in the sense of [6], and the correspondence $(\pi(\mathsf{P}), p) \mapsto (\pi(\overline{\mathsf{P}}), \overline{p})$ has a suitable universal property in 2-categorical terms.

As applications of this work, we provide an abstract method for models of *families of sets* for the extensional level of the *Minimalist foundation* [7], as done in [8] for the *Predicative effective topos* introduced in [9].

References

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