The algebraic weak factorisation system of twisted coreflections and delta lenses

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Abstract.

The notion of an algebraic weak factorisation system (AWFS), introduced by Grandis and Tholen [5], generalises that of an orthogonal factorisation system (OFS). In the standard definition of an AWFS on a category $\mathcal{C}$, the left and right classes of morphisms are determined by the categories of L-coalgebras and R-algebras for a suitable comonad-monad pair (L, R) on the arrow category $\mathcal{C}^{\mathbf{2}}$. An OFS may be understood as an AWFS in which the comonad and monad are idempotent. Recently, Bourke [2] demonstrated that an AWFS on $\mathcal{C}$ can be defined entirely in terms of a pair of double categories $\mathbb{L}$ and $\mathbb{R}$ over the double category of commutative squares $\mathbb{S}\mathrm{q}(\mathcal{C})$ equipped with a *lifting operation* that satisfies two axioms, yielding a characterisation much closer in spirit to that of an OFS.

A leading example of an algebraic weak factorisation system is the AWFS on $\mathcal{C}\mathrm{at}$ whose L-coalgebras are the split coreflections (functors equipped with a right-adjoint-left-inverse) and whose R-algebras are the split opfibrations. Delta lenses are functors equipped with a functorial choice of lifts, directly generalising the notion of split opfibration [6], and are the focus of ongoing research in applied category theory. Motivated by this close relationship with split opfibrations, it is natural to ask: is there an AWFS on $\mathcal{C}\mathrm{at}$ whose R-algebras are delta lenses?

In this talk, I will introduce the notion of *twisted coreflection* as a split coreflection with a certain property, and construct an algebraic weak factorisation system on $\mathcal{C}\mathrm{at}$ whose L-coalgebras are the twisted coreflections and whose R-algebras are the delta lenses.

I will present two separate descriptions of this AWFS, highlighting the connections between them. In the first approach [3], I will define explicitly the comonad L and monad R arising from a functorial factorisation on $\mathcal{C}\mathrm{at}$, in the sense of Grandis and Tholen [5]. In the second approach [4], I will construct double categories $\mathbb{T}\mathrm{wCoref}$ and $\mathbb{L}\mathrm{ens}$ of twisted coreflections and delta lenses, respectively, and define suitable lifts of twisted coreflections against delta lenses, in the sense of Bourke [2]. Both methods make important use the universal properties of bijective-on-objects functors, discrete categories, and the comprehensive factorisation system on $\mathcal{C}\mathrm{at}$. Moreover, I will show that this AWFS is cofibrantly generated by a small double category, in the sense of Bourke and Garner [1].

In addition to providing a new framework for understanding delta lenses, one of the principal benefits of this work is illustrating a seemingly rare example of a cofibrantly generated AWFS in which the entire left class, not just the generators, may be fully understood. In particular, I will show that every twisted coreflection arises as a pushout of an initial functor from a discrete category along a bijective-on-objects functor. This yields a simple way of constructing examples of twisted coreflections from indexed collections of categories with a chosen initial object. I will also give an explicit construction of the cofree twisted coreflection on a functor *and* on a split coreflection.

References

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