A higher-dimensional Eckmann–Hilton argument

A. S. Corner

Eugenia Cheng (info@eugeniacheng.com)
School of the Art Institute of Chicago

Alex Corner (alex.corner@shu.ac.uk)
Sheffield Hallam University

Abstract. The Eckmann–Hilton argument plays a subtle and crucial role behind the scenes in higher-dimensional category theory. It can show us where hidden commutativities arise from cells with identity boundaries being able to commute past each other, given enough dimensions. The basic case tells us that doubly-degenerate 2-categories give commutative monoids; more precisely, a doubly-degenerate 2-category "is" a set with two multiplications on it satisfying interchange, and the Eckmann–Hilton argument tells us that those multiplications must be the same and commutative [3].

Given more dimensions, more nuance is possible. It is considered well-known that doubly-degenerate weak 3-categories "are" categories with two weak monoidal structures satisfying weak interchange, and that a weak Eckmann–Hilton argument shows that this amounts to a braided monoidal category [4]. However, the classic proofs of this do not explicitly provide the generalisation of the Eckmann–Hilton argument [5].

At the next dimension the situation seems even more folkloric. It seems widely accepted that a category with three monoidal structures on it (satisfying appropriate interchange) "is" a symmetric monoidal category, and that this is the largest number of monoidal structures that will fit: adding further monoidal categories just gives us symmetric monoidal categories again. Often no indication of how this is achieved is given, only a reference made to Baez and Dolan's Stabilisation Hypothesis [2] or an appeal to the work of Joyal and Street [5].

All of this is in some sense well known, but we do not that think this has been precisely written down. Aguiar and Mahajan [1] give a comprehensive account of the related notion of n-monoidal category, but this still does not quite fit the particular nuance that we are interested in: that multiple monoidal structures, inherited from higher compositions and so interacting appropriately under interchange, will 'stabilise' to what is effectively a single symmetric monoidal structure.

In another sense, the work of Joyal and Street [5] claims that identifying those braidings which are symmetries provides an equivalence of 2-categories between symmetric monoidal categories and categories with a symmetric multiplication, but the details to establish this equivalence are not described explicitly.

Often an appeal is made to the Eckmann–Hilton argument in order to fill in missing details, but often nothing resembling such an argument is provided in full. In this talk we will give the 3-fold generalisation of the weak Eckmann–Hilton argument in 4 dimensions, that is, for 3-degenerate 4-categories. This will enable us to not only give a precise sense in which 3-degenerate 4-categories give rise to symmetric monoidal categories, but also to fill in the details of [5] to provide:

1. a generalisation to n-degenerate $(n+1)$-categories,
2. an algebraically nice 2-category of these, and
3. a proof that this 2-category is biequivalent to the 2-category of symmetric monoidal categories.

This enables us to finally remove the quotation marks around "is" in the statement

An n-degenerate $(n+1)$-category "is" a symmetric monoidal category

to give a satisfying, fully algebraic proof of one part of the stabilisation hypothesis.

References

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