Two developments of "Separability of The Second Kind"

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Abstract. The categorical notion of a separable functor was first given by Năstăsescu, Van den Bergh, and Van Oystaeyen in [12]. A functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ is said to be separable if the natural transformation on hom-sets induced by F can be split by a natural transformation P. This definition is constructed so that separable morphisms of rings correspond to the restriction of scalars being separable in the sense of [12]. Therefore, the study of separable functors is closely tied to that of adjoint pairs. In [13], Rafael gave conditions in terms of the unit and counit of an adjunction (F, G) for the functors F or G to be separable. This was further generalized to the notion of separability of the second kind, by Caenepeel and Militaru [5] : Suppose that we have functors $F : \mathcal{C} \longrightarrow \mathcal{D}$ and $I : \mathcal{C} \longrightarrow \mathcal{X}$. Then, the functor F is said to be I-separable if the natural transformation on hom-sets induced by F is split up to the natural transformation induced by I.

This talk will consist of two parts, both revolving around the notion of separability of the second kind. The first part is published work with Abhishek Banerjee [2], in which we bring together separability of the second kind and another notion of separable functors that has appeared in the literature : Heavily separable functors due to Ardizzoni and Menini [1] i.e., separable functors F such that the splitting natural transformation P is compatible with compositions in $\mathcal{D}$ in a certain manner. We combine these ideas to consider functors $F : \mathcal{C} \longrightarrow \mathcal{D}$ which are **heavily I-separable**, where I is a functor $I : \mathcal{C} \longrightarrow \mathcal{X}$.

We proceed to give a Rafael-type Theorem appropriate for this "amalgamated" version of separability. This characterizes heavy separability of the second kind for functors admitting a (left or right) adjoint. We then present applications of these results in three different contexts. The first application is in the context of ringoids, which horizontally categorify rings. These results generalize Ardizzoni and Menini's results on the ordinary heavy separability of functors associated with ring extensions.

The second application is to monads and comonads and the associated Eilenberg-Moore adjunctions. By fixing a category $\mathcal{C}$ and a monad $\mathbf{T}$ on $\mathcal{C}$, one can look at the family of $\mathbf{T}$-adjunctions, i.e., adjoint pairs $(F : \mathcal{C} \longrightarrow \mathcal{D}, G : \mathcal{D} \longrightarrow \mathcal{C})$ whose associated monad is $\mathbf{T}$. If $I : \mathcal{C} \longrightarrow \mathcal{X}$ is any functor and (F, G), (F', G') are $\mathbf{T}$-adjunctions, we show [2, § 4.] that the left adjoint F is heavily I-separable if and only if so is F'. This means that for a given monad $\mathbf{T}$, we can ask if the family of $\mathbf{T}$-adjunctions as a whole, is heavily I-separable. A dual result holds for comonads. These results are motivated by the work of Mesablishvili [10] with I-separability and families of adjunctions associated to a given monad or comonad. Now suppose that (L, R) is an adjunction such that the left adjoint L can be equipped with the structure of a comonad $\mathbf{L}$ on $\mathcal{C}$. It is known ([3, § 2.6]) that the right adjoint R

can be equipped with the structure of a monad $\mathbf{R}$. For any adjoint pair $(I \dashv J)$ of endofunctors on $\mathcal{C}$ satisfying a commutativity condition [2, Theorem 4.3.], we see that the free $\mathbf{L}$-coalgebra functor $F^{\mathbf{L}}$ taking objects of $\mathcal{C}$ to free $\mathbf{L}$-coalgebras is heavily I-separable if and only if the functor $F_{\mathbf{R}}$ taking objects of $\mathcal{C}$ to free $\mathbf{R}$-algebras is heavily J-separable. We then combine this with the results of Ardizzoni and Menini [1] to give two applications.

If time permits, we also see a third application of the Rafael-type Theorem in the context of entwined modules. This is one of the original contexts studied by Caenepeel and Militaru while introducing separability of the second kind in [5].

In the second part of the talk, we enrich the notion of separability of the second kind over a symmetric monoidal closed category $\mathcal{V}$. In particular, using Lawvere's remarkable idea of viewing metric spaces as enriched categories, we see that separability of the second kind yields a very simple geometric condtion when $\mathcal{V} = (([0,\infty], \geq), +, 0)$. We end with two theorems : First, we see an enriched version of the Rafael-type Theorem appropriate to this context. Second, we see that this notion of enriched separability is invariant under change of base.

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