Axioms for the category of finite-dimensional Hilbert spaces and linear contractions

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Abstract.
The category **Hilb** of Hilbert spaces and bounded linear maps and the category **Con** of Hilbert spaces and linear contractions were both recently characterised in terms of simple category-theoretic structures and properties [2, 3]. For example, the structure of a *dagger*—an involutive identity-on-objects contravariant endofunctor—encodes adjoints of linear maps. Remarkably, none of the axioms refer to analytic notions such as norms, continuity or real numbers.

Counterintuitively, characterising categories with only *finite-dimensional* Hilbert spaces is more challenging than those with *all* Hilbert spaces. The problem is that directed colimits are the natural categorical way to encode analytic completeness of the scalar field, but the existence of too many of these colimits also implies the existence of objects corresponding to infinite-dimensional spaces. In fact, to prove that the scalars are the real or complex numbers without such an infinite-dimensional object, appeal to Solèr's theorem [4] is no longer possible, so an entirely new approach is necessary.

This talk will introduce a characterisation, stated below, of the category **FCon** of *finite-dimensional* Hilbert spaces and linear contractions. It will focus on a new approach to proving that the scalars are the real or complex numbers that does not rely on the existence of infinite-dimensional objects. In this proof, the supremum of a bounded increasing sequence of positive scalars is explicitly constructed using the colimit of a directed diagram associated to the sequence. The talk will also briefly touch on the new notions of *bounded sequential diagram* and *dagger finiteness*, which are needed to address finite dimensionality. It is based on recent joint work with Chris Heunen [1].

Theorem. *A locally small dagger rig category* $(\mathbf{D}, \otimes, I, \oplus, O)$ *is equivalent to* **FCon** *if and only if*

(1) the object O is terminal (and thus a zero object), the canonical projections

$$p_1 = \big(X \oplus Y \xrightarrow{1 \oplus 0} X \oplus O \cong X\big) \qquad \text{and} \qquad p_2 = \big(X \oplus Y \xrightarrow{0 \oplus 1} O \oplus Y \cong Y\big)$$

are jointly monic, and there is a morphism $d \colon I \to I \oplus I$ such that $p_1 d \neq 0 \neq p_2 d$;

(2) the object I is dagger simple and a monoidal separator;

(3) every parallel pair has a dagger equaliser and every dagger monomorphism is a kernel;

(4) for all epimorphisms $x \colon A \to X$ and $y \colon A \to Y$, we have $x^\dagger x = y^\dagger y$ if and only if there is an isomorphism $f \colon X \to Y$ such that $y = fx$;

(5) every bounded sequential diagram has a colimit; and,

(6) every object is dagger finite.

References

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