Pushforward monads

A. Doña Mateo

Adrián Doña Mateo (adrian.dona@ed.ac.uk)
University of Edinburgh

Abstract.

If $\mathbf{T}$ is a monad on $\mathcal{C}$ and $G\colon \mathcal{C} \to \mathcal{D}$ is right adjoint to F, then $G\mathbf{T}F$ is a monad on $\mathcal{D}$, which we denote $G_{\#}\mathbf{T}$. Even when G is not a right adjoint, we can define $G_{\#}\mathbf{T}$ subject to the existence of a right Kan extension. This is the pushforward of $\mathbf{T}$ along G. This construction was first considered by Street in [2], where it takes place in a general (strict) 2-category, but has received very little attention since.

In this talk, I will review its definition, introduce its functoriality properties with respect to G and $\mathbf{T}$, and state the universal property satisfied by $G_{\#}\mathbf{T}$. Pushforwards turn out to be intimately related to codensity monads: pushing the identity monad forward along G gives the codensity monad of G. Moreover, any pushforward monad is a codensity monad, whereby $G_{\#}\mathbf{T}$ is the codensity monad of $GU^{\mathbf{T}}$, with $U^{\mathbf{T}}$ being the forgetful functor from the category of $\mathbf{T}$-algebras.

I will then present examples of the pushforward of three families of monads on the category of finite sets along the inclusion **FinSet** $\hookrightarrow$ **Set**. Each of these turn out to be related to the well-known codensity monad of this inclusion, which was shown to be the ultrafilter monad by Kennison and Gildenhuys [1]. Lastly, I will identify the category of algebras of the codensity monad of **Field** $\hookrightarrow$ **Ring** as the free product completion of **Field**, denoted **Prod**(**Field**). Pushing this monad forward along the forgetful functor **Ring** $\to$ **Set** gives the codensity monad of **Field** $\to$ **Set**. Its category of algebras is still **Prod**(**Field**), giving the remarkable fact that **Prod**(**Field**) is monadic over **Set**, even though **Field** is famously not.

References

[1] J. F. Kennison, and D. Gildenhuys, *Equational completion, model induced triples and pro-objects*, Journal of Pure and Applied Algebra 1.4 (1971).

[2] R. Street, *The formal theory of monads*, Journal of Pure and Applied Algebra 2.2 (1972).