Galois theory and homology in quasi-abelian functor categories

N. Egner

Nadja Egner (nadja.egner@uclouvain.be)
Université catholique de Louvain

Abstract.

In this presentation based on [2], I will consider the category $\mathscr{A}^{\mathbb{T}}$ of functors from a finite category $\mathbb{T}$ to a quasi-abelian category $\mathscr{A}$, and show that, for any replete full subcategory $\mathbb{S}$ of $\mathbb{T}$, the full subcategory $\mathscr{F}$ of $\mathscr{A}^{\mathbb{T}}$ with objects the functors $F : \mathbb{T} \to \mathscr{A}$ with $F(T) = 0$ for all $T \notin \mathbb{S}$ is a reflective and, moreover, torsion-free subcateogry of $\mathscr{A}^{\mathbb{T}}$. This implies that the corresponding Galois structure is admissible, and I will characterize the (higher) central extensions in $\mathscr{A}^{\mathbb{T}}$ with respect to $\mathscr{F}$ and the classes of regular epimorphisms in $\mathscr{A}^{\mathbb{T}}$ and $\mathscr{F}$, respectively. More precisely, for a regular epimorphism α in $\mathscr{A}^{\mathbb{T}}$, the following conditions are equivalent:

1. α is a central extension.
2. The kernel $\mathrm{Ker}(\alpha)$ of α lies in $\mathscr{F}$.
3. The T-component α_T is an isomorphism for all $T \notin \mathbb{S}$.

Furthermore, I will give generalized Hopf formulae for homology.

Instances of the pair $(\mathscr{A}^{\mathbb{T}}, \mathscr{F})$ are given by $(\mathrm{Arr}(\mathscr{A}), \mathscr{A})$, $(\mathrm{Arr}^2(\mathscr{A}), 2\text{-}\mathrm{Arr}(\mathscr{A}))$ and, more generally, $(\mathrm{Arr}^n(\mathscr{A}), n\text{-}\mathrm{Arr}(\mathscr{A}))$ for every $n \geq 1$, where n-$\mathrm{Arr}(\mathscr{A})$ denotes the category with objects the chain complexes in $\mathscr{A}$ of length n. Since $\mathscr{A}$ is assumed to be quasi-abelian, $\mathrm{Arr}^n(\mathscr{A})$ is equivalent to the category $\mathrm{Grpd}^n(\mathscr{A})$ of internal n-fold groupoids in $\mathscr{A}$ and n-$\mathrm{Arr}(\mathscr{A})$ is equivalent to the category n-$\mathrm{Grpd}(\mathscr{A})$ of internal n-groupoids in $\mathscr{A}$.

Let me shortly recall the notions of (higher) central extensions and generalized Hopf formulae for homology.

Categorical Galois theory

Categorical Galois theory, see e.g. [3], generalizes both classical Galois theory and the theory of central extensions of groups. A *Galois structure* Γ consists of an adjunction

$$\mathscr{C} \underset{\mathsf{U}}{\overset{\mathsf{F}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathscr{F}$$

with unit η, and classes $\mathscr{E}$ and $\mathscr{Z}$ of morphisms in $\mathscr{C}$ and $\mathscr{F}$, respectively, that satisfy certain conditions. For any object B in $\mathscr{C}$, this induces an adjunction

$$\mathscr{E}(B) \underset{\mathsf{U}^B}{\overset{\mathsf{F}^B}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathscr{Z}(\mathsf{F}(B)),$$

where $\mathscr{E}(B)$, also denoted by $\mathrm{Ext}(B)$, is the full subcategory of the slice category $\mathscr{C} \downarrow B$ with objects the morphisms in $\mathscr{E}$ with codomain B. These are called the *extensions* of B. Categorical Galois theory is concerned with the study of the full subcategory $\mathrm{CExt}(B)$ of $\mathrm{Ext}(B)$ with objects the central extensions of B. This notion is defined in two steps:

- An extension $f : A \to B$ is called *trivial* if it lies in the essential image of U^B.
- It is called *central* if it is 'locally' trivial, i.e., there exists a *monadic extension* $p : E \to B$ such that the pullback $p^*(f)$ of f along p is a trivial extension.

If the Galois structure Γ is *admissible*, the fundamental theorem of categorical Galois theory asserts that, for any monadic extension $p : E \to B$, there is a characterization of the extensions of B, whose pullback along p is a trivial extension, in terms of internal actions of the *Galois pregroupoid* $\mathrm{Gal}(E, p)$.

The central extensions with respect to the Galois structure Γ_{Ab} given by the adjunction

$$\mathrm{Grp} \underset{\mathrm{I}}{\overset{\mathrm{Ab}}{\underset{\leftarrow}{\overset{\rightarrow}{\perp}}}} \mathrm{Ab},$$

where Ab and I are the abelianization and inclusion functors, respectively, and $\mathscr{E}$ and $\mathscr{Z}$ are the classes of surjective group homomorphisms in Grp and Ab, respectively, recover exactly the classical central extension of groups.

Generalized Hopf formulae for homology

In certain cases, see e.g. [1], the full subcategory $\mathrm{CExt}(\mathscr{C})$ of central extensions in $\mathscr{C}$ of $\mathrm{Ext}(\mathscr{C})$ induces itself a Galois structure Γ_1 with adjunction

$$\mathrm{Ext}(\mathscr{C}) \underset{\mathsf{U}_1}{\overset{\mathsf{F}_1}{\underset{\leftarrow}{\overset{\rightarrow}{\perp}}}} \mathrm{CExt}(\mathscr{C})$$

with unit η^1, and with $\mathscr{E}^1$ and $\mathscr{Z}^1$ the classes of *double extensions* defined relatively to $\mathscr{E}$ and $\mathscr{Z}$, respectively. It turns out that the functor $[-]^1 : \mathrm{Ext}(\mathscr{C}) \to \mathrm{Ext}(\mathscr{C})$ given on objects by $[f] := \mathrm{Ker}(\eta^1_f)$, factors through $\mathscr{C}$, i.e., there exists a functor $[-]_1 : \mathrm{Ext}(\mathscr{C}) \to \mathscr{C}$ such that $[-]^1 = \iota^1 \circ [-]_1$, where ι^1 maps an object B to the extension $B \to 0$.

If $p : P \to B$ is a *$\mathscr{E}$-projective presentation* of B, the *second Hopf formula for homology* of B with respect to $\mathscr{F}$ is defined as

$$H_2(B, \mathscr{F}) := \tfrac{[P] \cap \mathrm{Ker}(p)}{[p]_1},$$

where $[P] := \mathrm{Ker}(\eta_P)$. More generally, the $(n+1)$-st Hopf formula for homology $H_{n+1}(B, \mathscr{F})$ is defined using the notions of n-fold central extensions and n-fold $\mathscr{E}$-projective presentations.

The generalized Hopf formulae with respect to the Galois structure Γ_{Ab} recover exactly the integral homology groups.

References

[1] T. Everaert, M. Gran and T. Van der Linden, *Higher Hopf formulae for Homology via Galois Theory*, Adv. in Mathematics 217 (2008), 2231–2267.

[2] N. Egner, *Galois theory and homology in quasi-abelian functor categories*, preprint arXiv:2403.12750, 2024.

[3] G. Janelidze, *Categorical Galois theory: revision and some recent developments*, Galois connections and applications, Math. Appl., vol. 565, Kluwer Acad. Publ., 2004, 139–171.