A monadic approach to non-commutative Stone duality

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Abstract. Boolean algebras axiomatise the theory of the set-theoretic operations $\cap$, $\cup$, $(\ )^c$ and $\varnothing$ on power-set lattices; Boolean semilattices do the same thing for the operations $\cap$, $\cup$, $\setminus$ and $\varnothing$; while the *left skew Boolean algebras* of Leech [4] do the same again for the operations $\vee, \wedge, \setminus$ and $\varnothing$ on sets of partial functions $\mathrm{Pfn}(X, Y)$ defined by:

$$f \wedge g = f|_{\mathrm{dom} f \cap \mathrm{dom} g}\,, \quad f \setminus g = f|_{\mathrm{dom} f \setminus \mathrm{dom} g}\,, \quad \varnothing = \text{undefined everywhere,} \qquad \text{and} \qquad f \vee g(x) = \begin{cases} g(x) & \text{if } x \in \mathrm{dom}(g) \\ f(x) & \text{if } x \in \mathrm{dom}(f) \setminus \mathrm{dom}(g) \\ \text{undefined} & \text{otherwise.} \end{cases}$$

The axioms resemble those for Boolean semilattices; the main difference is that $\vee$ and $\wedge$ are not commutative, so that left skew Boolean algebras are a non-commutative generalisation of Boolean semilattices.

As is well known, the category of Boolean algebras is contravariantly equivalent to the category of Stone spaces, i.e., totally disconnected compact Hausdorff spaces; this is Stone duality. A very mild generalisation of this shows that Boolean semilattices are equivalent to pointed Stone spaces. Much more far-reaching is the *non-commutative Stone duality* of Kudryavtseva [3], which shows that the category of left skew Boolean algebras is equivalent to the category of sheaves on pointed Stone spaces.

Kudryavtseva's result is extremely pretty, but proving it is delicate and requires a fair amount of calculation. In this talk I will describe a different approach to its establishment which, if it perhaps does not simplify the calculations much, at least serves to justify them from category-theoretic first principles. This approach reconstructs non-commutative Stone duality from an adjunction

$$\mathbf{SBA}^{\mathrm{op}} \underset{R}{\overset{L}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathbf{Poly}$$

between the category $\mathbf{SBA}$ of left skew Boolean algebras and the category $\mathbf{Poly} = \mathbf{Fam}(\mathbf{Set}^{\mathrm{op}})$ of polynomial endofunctors of $\mathbf{Set}$, induced by homming into the polynomial $\top \colon 1 \to 2$.

As with any adjunction, one can consider the induced monad T on $\mathbf{Poly}$, and the comparison functor $K \colon \mathbf{SBA}^{\mathrm{op}} \longrightarrow \mathsf{T}\text{-}\mathbf{Alg}$. The monad T turns out to be Ellerman's *ultrasheaf monad* [1], whose algebras were characterised by Kennison [2] as the category $\mathbf{ShvKH}$ of sheaves on compact Hausdorff spaces; and the adjunction $L \dashv R$ turns out to be of *descent type*, so that $K \colon \mathbf{SBA}^{\mathrm{op}} \longrightarrow \mathbf{ShvKH}$ is fully faithful. It is now simply a matter of characterising the image of K in order to reconstruct

Kudryavtseva's result. Part of the interest here is in establishing that $L \dashv R$ is indeed of descent type; one could simply calculate away, but instead we appeal to a general, and apparently new, result which provides sufficient conditions for an adjunction to be of descent type.

References

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[3] Kudryavtseva, G. A refinement of Stone duality to skew Boolean algebras. *Algebra Universalis 67*, 4 (2012), 397–416.

[4] Leech, J. Skew Boolean algebras. *Algebra Universalis 27*, 4 (1990), 497–506.