Pasting diagrams beyond acyclicity

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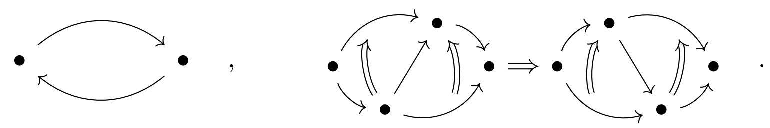
Abstract.
Most formalisms for n-categorical diagrams [1, 2, 4, 3] include some form of *acyclicity* condition on the shapes of diagrams. These conditions may be as strong as requiring the "flow" between cells across all dimensions to be acyclic, or as weak as only forbidding direct cycles when pasting along a single dimension. In either case, they forbid very simple non-pasting shapes already in dimension 1, and commonly occurring pasting shapes starting from dimension 3, such as



Moreover, acyclic shapes tend to not be stable under various useful constructions: typically, the stronger conditions are not stable under arbitrary duals, and the weaker conditions are not stable under pasting or under Gray products. A much better-behaved condition on shapes of diagrams is *regularity*: roughly, the requirement that all boundaries of all cells occurring in the diagram be, topologically, closed balls of the appropriate dimension.

The reason for the focus on acyclicity across all these sources is the insistence that the cells of the "presented ω-category" be *subsets* of cells of the diagram shape. In this talk, I will show that the problem disappears if one takes a functorial perspective, focussing on more general morphisms of diagram shapes. In particular, we can put ourselves in a convenient category $\mathbf{RDCpx}_{\downarrow}$ of *regular directed complexes*, which are poset-like structures encoding diagram shapes satisfying the regularity condition. Among the regular directed complexes, there is a class of objects, the *molecules*, which are shapes of pasting diagrams and satisfy a pasting theorem. Morphisms of $\mathbf{RDCpx}_{\downarrow}$ can be interpreted as "cellular" functors which are allowed to decrease the dimension of a cell; we let $\mathbf{RDCpx}_{=}$ be the wide subcategory whose morphisms are dimension-preserving.

The main result then states: *If P is a regular directed complex, the set $Mol/_P$ of isomorphism classes of objects $[f\colon U \to P]$ in the slice category $\mathbf{RDCpx}_{=}/P$ where U is a molecule admits a natural structure of strict ω-category. Moreover, this ω-category has a minimal generating set whose cells are in bijection with the elements of P.*

If, in addition, P satisfies a technical condition called *having frame-acyclic molecules*, then $Mol/_P$ is a polygraph (or computad). This is a much weaker acyclicity condition which, in particular, is always satisfied in dimension ≤ 3. Since regular directed complexes are closed under all sorts of useful operations, such as gluings (pushouts of monomorphisms), all duals, Gray products, suspensions, and joins, they provide a more convenient framework for n-categorical diagrams.

The content of this talk is based on parts of the upcoming monograph [5].

References

[1] M. Johnson. The combinatorics of n-categorical pasting. *Journal of Pure and Applied Algebra*, 62(3):211–225, 1989.

[2] R. Street. Parity complexes. *Cahiers de topologie et géométrie différentielle catégoriques*, 32(4):315–343, 1991.

[3] S. Forest. Unifying notions of pasting diagrams. *Higher Structures*, 6(1):1–79, 2022.

[4] R. Steiner. The algebra of directed complexes. *Applied Categorical Structures*, 1(3):247–284, 1993.

[5] A. Hadzihasanovic. Combinatorics of higher-categorical diagrams. In preparation, 2024.