# Virtual double categories as coloured box operads

L. Hermans

**Lander Hermans** (`lander.hermans@uantwerpen.be`)
University of Antwerp

**Abstract**.

Virtual double categories, also considered by Leinster as fc-multicategories [1], are a 2-categorification of multicategories and compare to double categories as multicategories compare to monoidal categories [2]. In algebraic topology, multicategories are also known as coloured operads and are extensively used to encode algebraic operations, thus generalizing operads.

In this talk we will generalize operads to box operads [3], fitting the following scheme:

| | | | | |
|---|---|---|---|---|
| operads | $\subset$ | multicategories = coloured operads | $\supset$ | monoidal categories |
| $\cap$ | | $\cap$ | | $\cap$ |
| box operads | $\subset$ | virtual double categories = coloured box operads | $\supset$ | double categories |

In particular, box operads correspond to virtual double categories with a single object, a single horizontal arrow and a single generating vertical arrow. We apply box operads to algebraic geometry and topology, as we now will explain.

We use a linear box operad $\mathsf{Lax}$ to encode lax functors over a small category taking values in linear categories. This is motivated by algebraic geometry where lax functors appear as prestacks generalizing structure sheaves and (noncommutative) deformations thereof.

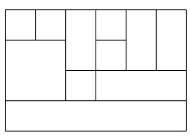
Our main results are the following:

1. The first operadic approach to an $L_\infty$-structure on the Gerstenhaber-Schack complex of a general prestack was given in [4]. Using box operads, in [3], we give explicit formulas in terms of stackings of rectangles ("boxes").

2. Making use of a newly developed *Koszul duality for box operads* which deals with non-quadratic relations, in [5], we establish a minimal (in particular cofibrant) model $\mathsf{Lax}_\infty$ of $\mathsf{Lax}$, shedding a new light on a question from Markl [6].

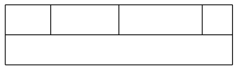
In this talk, we will mainly focus on sketching key components from (1) and (2) by harnessing a calculus of rectangles underlying box operads. In particular, we will present the following three results.

Box operads can be encoded as algebras over the symmetric coloured operad $\square\mathsf{p}$ (pronounced "boxop"). $\square\mathsf{p}$ consists of stackings of boxes which compose operadically by substituting a box in a stacking by a stacking of boxes. The following drawing provides an example of a stacking



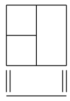
□p generalizes the well-known symmetric coloured operad Op encoding operads, which is defined using trees [7].

On the other hand, box operads can equivalently be encoded as monoids in a category with a *skew* monoidal product □ , called the "box composite". We will introduce the box composite using two-level stackings, for example



Interestingly, skew monoidal categories recently also appeared in various other contexts, such as operadic categories [8].

Thirdly, to each box operad we are able to associate higher algebraic operations $L_n$ constituting a $L_\infty$-algebra. A crucial role is played by *thin* boxes: boxes whose vertical sides are degenerate. The operations $L_n$ are obtained by summing over *thin-quadratic* stackings, i.e. stackings that are quadratic (in an appropriate sense) with respect to the thin boxes they contain. A thin-quadratic stacking that is not strictly quadratic is for example



Remark the bottom box is thin. This result generalizes the classical result for operads: quadratic trees induce a preLie-structure often called the Gerstenhaber brace.

Finally, I will explain briefly how these three ingredients play a key role in Koszul duality for box operads. If time permits, we will delve deeper by unpacking the (co)bar functor, twisting morphisms, the twisted complex and the application of Koszul duality to the box operad Lax.

Extending the above results to the coloured setting is an interesting topic for future research motivated by coloured versions of Lax. Part of this work is joint with Wendy Lowen and Hoang Dinh Van.

# References


[1] T. Leinster, fc-multicategories, preprint arXiv:9903004, 1999.

[2] Cruttwell, G. & Shulman, M. A unified framework for generalized multicategories. *Theory Appl. Categ.*. **24** pp. No. 21, 580-655, 2010.

[3] Dinh Van, H., Hermans, L. & Lowen, W. Box operads and higher Gerstenhaber brackets. preprint arXiv:2305.20036, 2023.

[4] Dinh Van, H., Hermans, L. & Lowen, W. Operadic structure on the Gerstenhaber-Schack complex for prestacks. *Selecta Math. (N.S.)*. **28**, Paper No. 47, 63, 2022.

[5] Hermans, L. A minimal model for prestacks via Koszul duality for box operads. preprint arXiv:2312.04185, 2023.

[6] Markl, M. Homotopy diagrams of algebras. *Proceedings Of The 21st Winter School "Geometry And Physics" (Srni, 2001).*, 161-180, 2002.

[7] Laan, P. Coloured Koszul duality and strongly homotopy operads. preprint arXiv:0312147, 2004.

[8] Lack, S. Operadic categories and their skew monoidal categories of collections. *High. Struct.*. **2**, 1-29, 2018.