Dagger category theory

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Abstract.

A *dagger* on a category associates to every morphism $f\colon A \to B$ a morphism $f^\dagger\colon B \to A$ going in the opposite direction in such a way that $f^{\dagger\dagger} = f$. Dagger categories are useful in many areas, including operator algebra, homological algebra, Bayesian inference, reversible computing, and quantum theory. Dagger categories can of course be studied using ordinary category theory. However, in many important ways, dagger categories behave very differently than ordinary categories. The situation compares to graph theory: directed and undirected graphs share a large part of theory, but many important results also distinguish them.

My goal in this talk is to convince you that *dagger category theory* is a very interesting area of study that relies on, but differs from, ordinary category theory. (For example, it is not just formal category theory in a universe other than **Cat** or enriched category theory over a base other than **Set**.)

We start by discussing examples. Any groupoid is an example of a dagger category, but $f^\dagger$ need not be the inverse of f; think about the transpose of a matrix, for example. The point is then made by showcasing three topics.

- The theory of *monads* works best when all structure respects the dagger: the monad and adjunctions should preserve the dagger. But for a smooth theory that is not enough. The monad and its algebras should should additionally satisfy the Frobenius law. Then any monad resolves as an adjunction, with extremal solutions given by the categories of Kleisli and Frobenius-Eilenberg-Moore algebras, which again have a dagger.
- There is a notion of *limit* for dagger categories that works well: it subsumes special cases such as dagger biproducts and dagger kernels; dagger limits are unique up to unique dagger isomorphism; a wide class of dagger limits can be built from a small selection of them; dagger limits of a fixed shape can be phrased as dagger adjoints to a diagonal functor. However, dagger categories with 'too many' dagger limits degenerate, and there is a more useful notion of dagger completeness.
- An important example is the dagger category of *Hilbert spaces*, with either continuous linear maps or linear contractions. In many ways it resembles the category of vector spaces, but it is not abelian, and the difference lies precisely in the dagger. We discuss a characterisation of this category by axioms that are elementary dagger-category-theoretic in nature and do not refer to analytic notions such as complex numbers, norm, continuity, convexity, or dimension.

References

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