On predicate liftings and lax extensions of functors

D. Hofmann

Dirk Hofmann (dirk@ua.pt)
University of Aveiro

Abstract.

Lax extensions of Set-functors to bicategories of (enriched) relations are a well-established tool in various parts of mathematics: they are fundamental in our work on "monoidal topology" [4], but also generic notions of bisimulation for coalgebras rely on lax extensions [5, 6]. Furthermore, by providing the semantical framework to interpret modal operators, predicate liftings are at the heart of the standard approach to coalgebraic modal logic [1].

One of the principal motivation for our work is the paper [6] where, among others, it is shown that

- for a normal (= identity preserving) lax extension L of a functor $F\colon \mathsf{Set} \to \mathsf{Set}$, L-bisimilarity captures precisely behavioural equivalence of F-coalgebras,
- the double-powerset functor does not have a lax extension L so that "L-bisimilarity captures behavioural equivalence", therefore the double-powerset functor does not have a normal lax extension,
- a finitary functor $F\colon \mathsf{Set} \to \mathsf{Set}$ has a normal lax extension if and only if F has a separating set of monotone predicate liftings.

Having these results as starting point, in this talk we will

- give necessary and sufficient conditions, in terms of (weak) preservation of certain pullbacks, for a Set-functor to admit a normal lax extension to Rel, as well as a largest normal lax extension,
- discuss uniqueness of normal lax extensions to Rel,
- provide a point-free perspective on the connection between lax extensions and predicate liftings in the context of quantale-enriched relations. In particular, we introduce a notion of predicate lifting for a lax extension which leads to a simple description of Moss lifting that goes beyond the realm of accessible functors and is independent of functor presentations (which feature centrally in previous approaches), and we show that every quantale-valued lax extension of an arbitrary Set-functor is induced by its class of Moss liftings [2]. We note here that this result explains the importance of the canonical extensions of generalized monotone neighborhood functors in the process of constructing quantale-valued lax extensions (in analogy to the two-valued case [6]); it is a stepping stone to connecting the coalgebraic approaches to behavioural distance via quantale-valued lax extensions and via liftings to categories of quantale-enriched categories, respectively [3].

This talk is based on joint work with Pedro Nora (University of Nijmegen), Sergey Goncharov, Lutz Schröder and Paul Wild (Friedrich-Alexander-Universität Erlangen-Nürnberg).

References

[1] C. Cîrstea, A. Kurz, D. Pattinson, L. Schröder, and Y. Venema, *Modal Logics are Coalgebraic*, The Computer Journal (2011), 54, 31–41.

[2] S. Goncharov, D. Hofmann, P. Nora, L. Schröder, and P. Wild, *A Point-free Perspective on Lax extensions and Predicate liftings*, Mathematical Structures in Computer Science (2024), 34, 1–30.

[3] S. Goncharov, D. Hofmann, P. Nora, L. Schröder, and P. Wild, *Kantorovich Functors and Characteristic Logics for Behavioural Distances*, in Foundations of Software Science and Computation Structures, O. Kupferman and P. Sobocinski, eds., vol. 13992 of Lecture Notes in Computer Science, Cham, 2023, Springer Nature Switzerland, pp. 46–67. 26th International Conference, FoSSaCS 2023, Held as Part of the European Joint Conferences on Theory and Practice of Software, ETAPS 2023, Paris, France, April 22–27, 2023, Proceedings.

[4] D. Hofmann, G. J. Seal, and W. Tholen, eds., *Monoidal Topology. A Categorical Approach to Order, Metric, and Topology*, vol. 153 of Encyclopedia of Mathematics and its Applications, Cambridge University Press, Cambridge, July 2014. Authors: Maria Manuel Clementino, Eva Colebunders, Dirk Hofmann, Robert Lowen, Rory Lucyshyn-Wright, Gavin J. Seal and Walter Tholen.

[5] J. Hughes and B. Jacobs, *Simulations in coalgebra*, Theoretical Computer Science (2004), 327, 71–108.

[6] J. Marti and Y. Venema, *Lax extensions of coalgebra functors and their logic*, Journal of Computer and System Sciences (2015), 81, 880–900.