Quotient toposes of discrete dynamical systems

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Abstract.

Lawvere's open problem on quotient toposes [1] has been solved for boolean Grothendieck toposes but not for non-boolean toposes. As a simple and non-trivial example of a non-boolean topos, we provide a complete classification of the quotient toposes of the topos of *discrete dynamical systems*. In this context, a discrete dynamical system means a pair (X, f) of a set X and an endofunction $f\colon X \to X$.

More concretely, our main theorem classifies the full subcategories of the topos $\mathbf{PSh}(\mathbb{N})$ that are closed under finite limits and small colimits. There are numerous such full subcategories, including those for which:

- Every state is in a loop.

$$\forall x \in X,\ \exists n > 0,\ f^n(x) = x.$$

- Every state is eventually fixed.

$$\forall x \in X,\ \exists n > 0,\ f^{n+1}(x) = f^n(x).$$

- f is bijective.
- Every state enters a loop within two steps, where the period of the loop has no square factors.

$$\forall x \in X,\ \exists n > 0,\ (f^{n+2}(x) = f^2(x)) \wedge (\forall p : \text{prime } p^2 \nmid n)$$

The goal of this talk is to describe these classes uniformly and clarify the background mathematical structures.

Our result is deeply related to monoid epimorphisms. Utilizing the theory of lax epimorphisms in the 2-category **Cat**, we will explain how (non-surjective) monoid epimorphisms from $\mathbb{N}$ correspond to (non-periodic) behaviors in discrete dynamical systems.

This talk is based on the joint work with Yuhi Kamio [2].

References

[1] W. Lawvere, *Open problems in topos theory*, 2009. (One can find it via nLab. `https://ncatlab.org/nlab/show/William+Lawvere`)

[2] R. Hora, and Y. Kamio *Quotient toposes of discrete dynamical systems*, Journal of Pure and Applied Algebra, accepted 2024, in press.