The Elementary Theory of the 2-Category of Small Categories

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Abstract.

Lawvere's *Elementary Theory of the Category of Sets* (ETCS) [3] posits that the category **Set** is a well-pointed elementary topos with natural numbers object satisfying the axiom of choice. This provides a category theoretic foundation for mathematics which axiomatises the properties of function composition in contrast to Zermelo-Fraenkel set theory with the axiom of choice (ZFC), which axiomatises sets and their membership relation. Furthermore, ETCS augmented with the axiom schema of replacement can be shown to be equiconsistent with ZFC.

In this talk, I will present a categorification of ETCS which axiomatises the 2-category of small categories, functors and natural transformations; this is *the elementary theory of the* 2*-category of small categories* (ET2CSC) of the title. This extends Bourke's [1] characterisation of categories internal to a category $\mathcal{E}$ with pullbacks to the setting where $\mathcal{E}$ satisfies the extra properties of ETCS. Important 2-categorical definitions I will introduce are 2-well-pointedness, the full subobject classifier and the categorified axiom of choice. The main conclusion is that ET2CSC is 'Morita biequivalent' with ETCS, meaning that the two theories have biequivalent 2-categories of models. The proof of this uses various adjunctions between $\mathcal{E}$ and $\mathbf{Cat}(\mathcal{E})$ in order to transfer properties from one to the other.

I will also describe how Shulman and Weber's ideas on discrete opfibration classifiers can be used to incorporate replacement, in a way reminiscent of algebraic set theory.

This talk is based on joint work with Adrian Miranda [2].

References

[1] John Bourke. *Codescent objects in 2-dimensional universal algebra.* PhD thesis, University of Sydney, 2010.

[2] C. Hughes, A. Miranda, *The Elementary Theory of the* 2*-Category of Small Categories*, preprint arXiv:2403.03647, 2024.

[3] F William Lawvere. An elementary theory of the category of sets. *Proceedings of the National Academy of Sciences*, 52(6):1506–1511, 1964.

[4] M. Shulman, Classifying Discrete Opfibration page on ncatlab. `https://ncatlab.org/michaelshulman/show/classifying+discrete+opfibration`, (Accessed Nov 2023, Revision 15).