An Enriched Small Object Argument Over a Cofibrantly Generated Base

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Abstract.

The small object argument is a transfinite construction of weak factorization systems developed by Quillen [1], originally motivated by homotopy theory. Since then, various variations ([2, 1.37], [3], [4, 13.2.1]) of the small object argument became an important tool in category theory itself and also in other fields of mathematics such as model theory due to the connection between the argument and ubiquitous notions of injectivity and orthogonality.

In the talk I will tell you about an enriched variant of the small object argument that subsumes the classical 1-categorical small object argument for weak factorization systems, the 1-categorical small object argument for orthogonal factorization systems, and certain variants of the small object argument for 2-categories, (2,1)-categories, dg-categories and simplicially enriched categories.

Along the way, we will introduce a variation of the Day convolution in which we use copowers instead of the monoidal product. In more detail: Given a cosmos $\mathcal{V}$ and a $\mathcal{V}$-category $\mathcal{K}$, we introduce an analogue $F * X\colon \mathcal{A} \to \mathcal{K}$ of the Day convolution for $\mathcal{V}$-functors $F\colon \mathcal{A} \to \mathcal{V}$, $X\colon \mathcal{A} \to \mathcal{K}$, in which we use copowers in $\mathcal{K}$ instead of the monoidal product in $\mathcal{V}$. This then makes the underlying category $[\mathcal{A}, \mathcal{K}]_0$ of the $\mathcal{V}$-category $[\mathcal{A}, \mathcal{K}]$ of $\mathcal{V}$-functors $\mathcal{A} \to \mathcal{K}$ a copowered $[\mathcal{A}, \mathcal{V}]$-category. These copowers play a central role in our variant of the small object argument.

The talk will be based on the preprint [5].

References

[1] D.G. Quillen, *Homotopical Algebra*, Lecture Notes in Mathematics, No. 43 Springer-Verlag, Berlin-New York, 1967.

[2] J. Adámek, J. Rosický, *Locally presentable and accessible categories*, London Mathematical Society Lecture Note Series, 189. Cambridge University Press, Cambridge, 1994.

[3] R. Garner, *Understanding the small object argument*, Appl. Categ. Structures 17 (2009), no. 3, 247–285.

[4] E. Riehl, *Categorical homotopy theory*, New Mathematical Monographs, 24. Cambridge University Press, Cambridge, 2014.

[5] J. Jurka, An Enriched Small Object Argument Over a Cofibrantly Generated Base, preprint arXiv:2401.05974, 2024.