On eigen-ring construction for monads

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Abstract.

For a ring A and its left ideal J, the eigen-ring [1] is defined by the quotient of the idealizer by J where the idealizer is the maximal subring of A which contains J as a two-sided ideal. Let k be a commutative unital ring. In this talk, we give a generalization of this concept by replacing rings with monads in the bicategory B_k whose objects are sets, morphisms are bi-indexed k-modules and 2-morphisms are intertwiners. This provides a uniform framework to understand some representations of categories which we explain below. As a fundamental result, for a monad $\mathtt{T}$, a left ideal $\mathtt{J} \subset \mathtt{T}$ and its eigen-ring $\mathrm{E}_{\mathtt{T}}(\mathtt{J})$, we give an adjunction between the category of $\mathtt{T}$-modules and the category of $\mathrm{E}_{\mathtt{T}}(\mathtt{J})$-modules. This adjunction is a generalization of the Morita equivalence between k-modules and modules over the matrix algebra.

Monads in B_k are equivalent with k-linear categories. Let $\mathtt{A}_{\mathcal{C}}$ be the monad corresponding to the k-linearization $k\mathcal{C}$ of a category $\mathcal{C}$. The purpose of this talk is to give specific left ideals which encode some properties of $\mathtt{A}_{\mathcal{C}}$-modules: to be precise, the category of $\mathtt{J}$-generated $\mathtt{A}_{\mathcal{C}}$-modules, which should be explained in this talk, is equivalent to the category of $\mathtt{A}_{\mathcal{C}}$-modules subject to that property. For example, if $\mathcal{C}$ is the opposite category gr° of finitely generated free groups, then the properties such as analyticity, polynomiality [3, 4, 6], outer property [5] and primitivity [2] correspond to certain left ideals $\mathtt{I}^{\nu}, \mathtt{I}^{d+1}, \mathtt{I}^{\mathsf{out}}, \mathtt{I}^{\mathsf{pr}}$ respectively. As one of our main results, the table below computes their eigen-rings where $\mathtt{P}^{d}$ is the monad related with augmentation ideals; $\mathtt{D}_{\mathfrak{Lie}}$ is the monad associated with Lie operad; $\mathtt{H}_0$ is the monad induced by the 0-th group homology of free groups. Moreover, the application of the above adjunction to each case leads to well-known adjunctions in the literature. In particular, the case of primitivity reproduces the universal enveloping algebra construction (more generally, Powell's construction [6]). This work is now in progress.

Monad $\mathtt{T}$	Property	Left ideal $\mathtt{J}$	Eigen-monad $\mathrm{E}_{\mathtt{T}}(\mathtt{J})$
$\mathtt{A}_{\mathsf{gr}^{\circ}}$	polynomial with degree $\leq d$	$\mathtt{I}^{d+1}$	$\mathtt{P}^{0}/\mathtt{P}^{d+1}$
	analytic	$\mathtt{I}^{\nu}$	$\mathrm{E}_{\mathtt{A}_{\mathcal{C}}}(\mathtt{I}^{\nu}_{\mathcal{C}})$
	primitive	$\mathtt{I}^{\mathsf{pr}}$	$\mathtt{D}_{\mathfrak{Lie}}$
	outer	$\mathtt{I}^{\mathsf{out}}$	$\mathtt{H}_0$

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