Limits in Enhanced Simplical Categories

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Abstract.

In [3], Riehl and Verity have developed the theory of ∞-*cosmoi*, which are quasi-categorically enriched categories that satisfy certain nice properties resembling enriched categories of fibrant objects. The theory of ∞-cosmoi provides a setting for understanding models for $(\infty, 1)$-categories with structures, and the pseudo morphisms between them. For instance, we have the ∞-cosmos of $(\infty, 1)$-categories admitting limits of shape J for a simplicial set J and the functors which preserve limits, and also the ∞-cosmos of Cartesian fibrations between $(\infty, 1)$-categories and the Cartesian functors.

Riehl and Verity have established in [3] that ∞-cosmoi admit all *flexible weighted limits*, which are simplicially enriched limits that are analogous to *PIE limits* in 2-category theory. Examples include products, inserters, and comma objects.

Besides, in [1], Lack has shown that the existence of 2-dimensional limits involving *lax* morphisms is subtle. For instance, in the 2-category of categories with limits of shape J and the functors that do *not* necessarily preserve limits, comma objects exist only when one of the 1-morphisms in the diagram preserves limits.

This phenomenon led Lack and Shulman to introduce *enhanced* 2-*category theory* in [2]. An enhanced 2-category is a 2-category with *two* types of 1-morphisms: the tight ones and the loose ones, in which every tight 1-morphism is also loose. For example, categories admitting J-shaped limits form an enhanced 2-category, with the tight 1-morphisms given by the functors that preserve limits, whereas the loose 1-morphisms given by just the functors, and the 2-morphisms given by the natural transformations. As enhanced 2-categories can be seen as enriched categories, Lack and Shulman have studied the subtle phenomenon of the existence of 2-dimensional limits involving *lax* morphisms via enriched category theory.

Taking inspiration from the work [2] by Lack and Shulman, we introduce the notion of *enhanced simplicial categories*, which are basically simplicial categories with *two* types of 0-arrows. Similarly, an enhanced simplicial category can be seen as an enriched category, hence we apply enriched category theory to study limits for *lax* morphisms in the ∞-categorical setting.

In the talk, we show that many interesting enhanced simplicial categories, such as that of $(\infty, 1)$-categories possessing limits together with the pseudo and lax morphisms between them, admit certain weighted limits. These include comma objects with one tight 0-arrow in the diagram, ∞-categorical versions of equifiers and inserters, and some further advanced limits, all involving loose 0-arrows in the corresponding diagrams. In particular, these results specialise to any model for $(\infty, 1)$-categories, generalising results on quasi-categories and also categories.

References

[1] S. Lack, *Limits for lax morphisms*, Appl. Categ. Structures 13 (2005), no. 3, 189–203.

[2] S. Lack, and M. Shulman, *Enhanced 2-categories and limits for lax morphismss*, Adv. Math. 229 (2012), no. 1, 294–356.

[3] E. Riehl, and D. Verity, *Elements of ∞-Category Theory*, first ed., Cambridge Studies in Advanced Mathematics, vol. 194, Cambridge University Press, 2022.