The Grothendieck construction in the context of tangent categories

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Abstract.

Cockett and Cruttwell in their investigation of vector bundles in the context of a tangent category (cf. [3]), which led to the concept of differential bundles, came out with an important generalization of the notion of fibrations: tangent fibrations. In a nutshell, a tangent fibration is a fibration between two tangent categories, which is also a strict tangent morphism. Cockett and Cruttwell also realized that, by pulling back along the zero morphism of the base tangent structure, the fibres of a tangent fibration inherit a tangent structure. So, a tangent fibration can be sent to an indexed tangent category, which is an indexed category whose fibres have a tangent structure, in a compatible way with the indexing and the substitution functors between them.

Famously, the Grothendieck construction establishes an equivalence between (cloven) fibrations and indexed categories (cf. [2]), so it is natural to wonder whether or not the operation introduced by Cockett and Cruttwell which sends a tangent fibration to its indexed tangent category can be extended to an equivalence. In particular, the question is whether or not we can reconstruct the total tangent structure, i.e. the tangent structure over the total category, of the tangent fibration, starting from the associated indexed tangent category.

The answer is only partial: by pulling back via the zero morphism we inevitably lose some information about the total tangent structure. To solve this issue I explored a generalization of tangent structures: the notion of tangent objects. A tangent object consists of an object in a given 2-category together with a tangent structure on it. It is important to mention that in his thesis, Leung went close to introducing this concept, by generalizing tangent structures over an arbitrary monoidal category (see [4]. In my version, I generalize this notion over a 2-category instead) and Bauer, Burke, and Ching employed a similar idea to introduce tangent ∞-categories in [1].

This simple generalization of a tangent category leads to interesting questions and new approaches to the theory of tangent categories. In particular, tangent objects in the 2-category of fibrations over a non-fixed base category are precisely tangent fibrations. From this observation, I proved the main result: tangent fibrations are equivalent to tangent indexed categories (to not be confused with indexed tangent categories mentioned earlier), which are tangent objects in the 2-category of indexed categories over a non-fixed indexing category.

In my talk, I would like to briefly recall the definition of a fibration, of an indexed category, and the Grothendieck construction in the classical case. Then, I would like to introduce the notion of tangent fibrations, as presented by Cockett and Cruttwell, and briefly discuss their result which leads to the notion of indexed tangent categories. I plan to discuss how to partially reconstruct the original tangent fibration from the associated indexed tangent category and show what fails to be recovered.

I will dedicate some time to introduce the main new technology: tangent objects, discuss a few important examples, like tangent categories, which are tangent objects in the 2-category of categories, tangent monads, which turn out to be tangent objects in the 2-category of monads, and finally tangent fibrations, as tangent objects in the 2-category of fibrations. Finally, I would like to unpack the definition-theorem of the equivalence between tangent fibrations and the new notion of tangent indexed categories.

My paper is available on Arxiv here:
https://arxiv.org/abs/2311.14643

References

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[3] R. Cockett, G. Cruttwell, *Differential Bundles and Fibrations for Tangent Categories*, Cahiers de Topologie et Géométrie Différentielle Catégoriques (2018), vol. LIX, 10–92.

[4] P. Leung, *Tangent bundles, monoidal theories and Weil algebras*, Bulletin of the Australian Mathematical Society, vol. 98 no. 1, 2018.