Identity objects and virtualisation

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Abstract.

In categorical logic, indexed preorders are an interpretation of many-sorted predicate logic. Taking the view that many-sorted predicate logic is a highly truncated version of dependent type theory, we obtain the following adaptation of the inductive axioms of identity types [1] to indexed preorders.

Definition. Let $P^1 \colon (P^0)^{\mathrm{op}} \to \mathrm{Pre}^{\wedge,\top}$ be an indexed $(\wedge,\top)$-preorder over a binary-product category P^0. An *identity object* on an object $X \in P^0$ is an element $\mathrm{Id}_X \in P^1(X \times X)$, such that

1. (*introduction* or *reflexivity*) $\top \leq (X \overset{\delta}{\to} X \times X)^*(\mathrm{Id}_X)$, and
2. (*elimination*) for any object $Y \in P^0$ and $p, q \in P^1(X \times X \times Y)$, if

$$(X \times Y \overset{\delta \times Y}{\to} X \times X \times Y)^*(p) \leq (X \times Y \overset{\delta \times Y}{\to} X \times X \times Y)^*(q),$$

   then $(X \times X \times Y \overset{\pi_1;\pi_2}{\to} X \times X)^*(\mathrm{Id}_X) \wedge p \leq q$.

We say $P \coloneqq (P^0, P^1)$ *has identity objects* if each X has an identity object.

This Martin-Löf notion of equality turns out, perhaps as expected, to be equivalent to Lawvere's one as extracted by Maietti and Rosolini in the notion of elementary doctrine [2]:

Theorem. *An indexed $(\wedge,\top)$-poset over a finite-product category has identity objects if and only if it is an elementary doctrine.*

This means Pasquali's 'elementary completion' result [4] is telling us that the *equivalence relations construction* $P \mapsto \mathrm{ER}(P)$ underlying Maietti and Rosolini's 'effective-quotient completion' [3] is a right-biadjoint completion that adds identity objects. Pasquali's result adapted to our settings reads:

Theorem. *The assignment $P \mapsto \mathrm{ER}(P)$ extends to a 2-functor* $\mathrm{IdxPre}_{\mathrm{pn}}^{\times,\wedge,\top} \to \mathrm{IdxPre}_{\mathrm{pn}}^{\times,\wedge,\top,\mathrm{Id}}$ *that is right biadjoint to the inclusion 2-functor.*

Here, the notation e.g. $\mathrm{IdxPre}_{\mathrm{pn}}^{\times,\wedge,\top,\mathrm{Id}}$ denotes the 2-category of indexed $(\wedge,\top)$-preorders with identity objects over binary-product categories, **p**seudo**n**atural morphisms that preserves $\times$, $\wedge$, $\top$ and Id, and 2-morphisms; these morphisms and 2-morphisms are defined in the same way as in [2, 3, 4], except that our morphisms have a *pseudo*natural-transformation component.

We produce an analogue of this result for the *PER construction*, the partial equivalence relations version of the ER construction, which appears as a key step in the tripos-to-topos construction [5].

Let P be an indexed $\wedge$-preorder over a binary-product category. The PER construction is given by an indexed preorder $\mathrm{PER}(P)$ defined in the same way as $\mathrm{ER}(P)$ but with as objects in $\mathrm{PER}(P)^0$ *partial* equivalence relations in P instead. Now the following weakened form of identity objects allows us to render the PER construction as a right-biadjoint completion.

Definition. We say P *has partial identity objects* if each object $X \in P^0$ is *equipped* with an element $\mathrm{PId}_X \in P^1(X \times X)$, such that

1. (*partial reflexivity*) $\mathrm{PId}_X \le (X \times X \overset{\pi_1}{\to} X \times X)^*(\mathrm{PId}_X), (X \times X \overset{\pi_2}{\to} X \times X)^*(\mathrm{PId}_X)$,
2. (*paravirtual elimination*) for each object $Y \in P^0$ and elements $p, q \in P^1(X \times X \times Y)$, if
$$(X \times Y \overset{\pi_1;\pi_1}{\to} X \times X)^*(\mathrm{PId}_X) \wedge (X \times Y \overset{\pi_2;\pi_2}{\to} Y \times Y)^*(\mathrm{PId}_Y) \wedge$$
$$(X \times Y \overset{\delta\times Y}{\to} X \times X \times Y)^*(p) \le (X \times Y \overset{\delta\times Y}{\to} X \times X \times Y)^*(q),$$
then $(X \times X \times Y \overset{\pi_3;\pi_3}{\to} Y \times Y)^*(\mathrm{PId}_Y) \wedge (X \times X \times Y \overset{\pi_1;\pi_2}{\to} X \times X)^*(\mathrm{PId}_X) \wedge p \le q$,
3. each arrow $f\colon X \to Y$ in P^0 satisfies $\mathrm{PId}_X \le (f \times f)^*(\mathrm{PId}_Y)$, and
4. $\mathrm{PId}_{X\times Y} \simeq (X \times Y \times X \times Y \overset{\pi_1;\pi_3}{\to} X \times X)^*(\mathrm{PId}_X) \wedge (X \times Y \times X \times Y \overset{\pi_2;\pi_4}{\to} Y \times Y)^*(\mathrm{PId}_Y)$.

Theorem. *The assignment* $P \mapsto \mathrm{PER}(P)$ *extends to a 2-functor* $\mathrm{IdxPre}_{\mathrm{pn}}^{\times,\wedge} \to \mathrm{IdxPre}_{\mathrm{pn}}^{\times,\wedge,\mathrm{PId}}$ *that is right biadjoint to the forgetful 2-functor.*

Indexed preorders with partial identity objects can be promoted to those with identity objects by a construction we call *virtualisation.* This is in fact another step in the tripos-to-topos construction, which turns the PERs into equivalence relations. An indexed preorder P is *oplaxly sectioned* if each object $X \in P^0$ is equipped with an element $\mathrm{os}_X \in P^1(X)$, and every arrow $f\colon X \to Y$ in P^0 satisfies $\mathrm{os}_X \le f^*(\mathrm{os}_Y)$. We regard an indexed preorder with partial identity objects as oplaxly sectioned, by $\mathrm{os}_X \coloneqq (X \overset{\delta}{\to} X \times X)^*(\mathrm{PId}_X)$. Let P be an oplaxly sectioned indexed $\wedge$-preorder.

Definition. The *virtualisation* of P is the indexed preorder $\mathrm{Virt}(P)$ given by $\mathrm{Virt}(P)^0 \coloneqq P^0$ and $\mathrm{Virt}(P)^1(X) \coloneqq (\mathrm{U}_{\mathrm{Set}} P^1(X), \overset{\mathrm{v}}{\le})$ where $p \overset{\mathrm{v}}{\le} q$ if and only if $\mathrm{os}_X \wedge p \le q$.

$\mathrm{Virt}(P)^1$ is in fact a Kleisli as well as Eilenberg-Moore object for a (necessarily idempotent) comonad in the Pre-enriched category $[(P^0)^{\mathrm{op}}, \mathrm{Pre}^{\wedge}]_{\mathrm{o}}$ of functors and *oplax* natural transformations.

Note that the os_X become top elements in $\mathrm{Virt}(P)$, and if P has partial identity objects, then the PId_X become identity objects in $\mathrm{Virt}(P)$. Virtualisation has the following universal properties; beware that mainly **o**plax-**n**atural morphisms are involved here, rather than pseudonatural morphisms.

Theorem. *The assignment* $P \mapsto \mathrm{Virt}(P)$ *extends to a 2-functor* $\mathrm{IdxPre}_{\mathrm{on}}^{\wedge,\mathrm{os}} \to \mathrm{IdxPre}_{\mathrm{on}}^{\wedge,\top}$ *as well as a 2-functor* $\mathrm{IdxPre}_{\mathrm{on}}^{\times,\wedge,\mathrm{PId}} \to \mathrm{IdxPre}_{\mathrm{on}}^{\times,\wedge,\top,\mathrm{Id}}$ *that is ambidextrously biadjoint to 'the' respective inclusion 2-functor. The left-biadjoint part also holds with respect to pseudonatural morphisms.*

References

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[4] F. Pasquali, *A Co-free Construction for Elementary Doctrines*, App. Cat. Str. 23 (2015), pp. 29–41.

[5] J.M.E. Hyland, P.T. Johnstone and A.M. Pitts, *Tripos theory.* Math. Proc. Camb. Philos. Soc. 88 (1980), no. 2, pp. 205–232.