Free Differential Storage Modalities

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Abstract.
Storage modalities provide the categorical interpretation of the exponential modality from Linear Logic, while differential storage modalities [1, 2] do the same in Differential Linear Logic. Briefly, a storage modality on a symmetric monoidal category with finite products is a comonad ! such that every $!A$ is naturally a cocommutative comonoid and we have the Seely isomorphism $!(A \times B) \cong !A \otimes !B$. A differential storage modality on an additive symmetric monoidal category with finite biproducts is a storage modality ! which comes equipped with a natural transformation $\mathsf{d} : !A \otimes A \to !A$, called the deriving transformation, whose axioms are based on the fundamental identities of differentiation such as the product rule and the chain rule. Using Kelly's notion of algebraically-free commutative monoids [3], we construct free differential storage modalities over storage modalities. A symmetric monoidal category is said to be endowed with algebraically-free commutative monoids if for every object X, there is an object $\mathsf{S}(X)$ equipped with a map $\mathsf{S}(X) \otimes X \to \mathsf{S}(X)$ which is universal amongst commutative right X-actions $A \otimes X \to X$. Then for an additive symmetric monoidal category with finite biproducts which is endowed with algebraically-free commutative monoids, for every storage modality !, we get that $!(-) \otimes \mathsf{S}(-)$ is the free differential storage modality over !. In other words, in this setting, the forgetful functor from the category of differential storage modalities to the category of storage modalities has a left adjoint. Moreover, when taking ! to be the initial storage modality, we get the initial differential storage modality which is related to the Faà di Bruno construction [2] and also recaptures the exponential modality in Clift and Murfet's Differential Linear Logic model [4].

References

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