Homotopy colimits enriched over a general base

G. Leoncini

Giuseppe Leoncini (leoncini@math.muni.cz)
Masaryk University

Abstract.

Starting from a 1-categorical base $\mathcal{V}$ which is *not* assumed endowed with a choice of model structure (or any kind of homotopical structure), we define homotopy colimits enriched in $\mathcal{V}$ in such a way that, for $\mathcal{V} = \mathsf{Set}$, we retrieve the classical theory as presented in [1] and [3]. We construct the free homotopy $\mathcal{V}$-cocompletion of a small $\mathcal{V}$-category and show that it satisfies the expected universal property. For $\mathcal{V} = \mathsf{Set}$, we retrieve Dugger's construction of the universal homotopy theory on a small category $\mathcal{C}$. We define the homotopy theory of internal ∞-groupoids in $\mathcal{V}$ as the homotopy $\mathcal{V}$-cocompletion of a point, and argue that $\mathcal{V}$-enriched homotopy colimits correspond to colimits in ∞-categories enriched in internal ∞-groupoids in $\mathcal{V}$, thus providing a convenient model to perform computations. Again, taking $\mathcal{V} = \mathsf{Set}$, this retrieves the classical notions for ordinary $(\infty, 1)$-categories. We compare our approach with some previous definitions of enriched homotopy colimits, such as those in [4] and [6]. As an application, we settle, for any group, a conjecture that in the case of a finite group was recently proven by completely different methods in [5]: we show that the so-called genuine homotopy theory of G-spaces is the G-equivariant homotopy cocompletion of a point. We conclude providing further examples of homotopy theories that can be seen as homotopy $\mathcal{V}$-cocompletions for a suitable choice of enrichment.

References

[1] W. Dwyer, P. S. Hirschhorn, D. Kan, J. Smith, *Homotopy Limit Functors on Model Categories and Homotopical Categories*, Mathematical Surveys and Monographs, Volume 113, American Mathematical Society, 2004.

[2] D. Dugger, *Universal Homotopy Theories*, Advances in Mathematics 164, 2001, pp. 144-176.

[3] P. S. Hirschhorn, *Model categories and their localizations*, Mathematical Surveys and Monographs, volume 99, American Mathematical Society, 2003.

[4] S. Lack and J. Rosický, *Homotopy locally presentable enriched categories*, Theory and Applications of Categories, Vol. 31, No. 25, 2016, pp. 712–754.

[5] J. Shah, *Parametrized higher category theory*, Algebraic and Geometric Topology 23, 2023, 509–644.

[6] M. Shulman, *Homotopy limits and colimits and enriched homotopy theory*, arXiv:math/0610194, 2009.