Logical Structure in (Homotopical) Inverse Functor Categories

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Abstract.

Reedy categories have been the subject of interest in categorical homotopy theory due to their ability to give an explicit, yet general, construction of the model structure on the category of simplicial objects in a model category. Inverse categories, which are the special case of Reedy categories where all (non-invertible) maps lower degree, have also been of interest: for instance, in type theory, [3] used them to provide a partial solution to the homotopy canonicity hypothesis, while [1] used them to put a model structure on the category of models of type theory. In both of these cases, one shows that if $\mathcal{E}$ is a model of type theory and $\mathcal{I}$ is an inverse category then $\mathcal{E}^{\mathcal{I}}$ forms a model of type theory as well. In [2], it is further shown that the category of homotopical diagrams $\mathcal{E}^{\mathcal{I}^{-1}\mathcal{I}}$ (that is, the full subcategory of functors $\mathcal{I} \to \mathcal{E}$ inverting all maps) is closed under much of the type-theoretic logical structure of $\mathcal{E}^{\mathcal{I}}$. While this result is sufficient for most type-theoretic considerations, it is natural to consider generalisations when $\mathcal{I}$ is equipped with weak equivalences $\mathcal{W}$. In this talk, we further consider the following questions and provide affirmative answers to them.

- Can one find conditions on $\mathcal{W}$ such that $\gamma^* \colon \mathcal{E}^{\mathcal{W}^{-1}\mathcal{I}} \to \mathcal{E}^{\mathcal{I}}$ is ensured to preserve the logical structure? In particular, as the techniques of [2] used to show the preservation of dependent products rely heavily on the assumption that all maps are inverted in $\mathcal{I}$, can one weaken this condition?
- Can one isolate the type-theoretic aspects from the logical aspects? For example, can one show that if $\mathcal{E}$ is a topos then so is $\mathcal{E}^{\mathcal{I}}$?

Here, we take advantage of the inverse structure of $\mathcal{I}$ to prove, for a category $\mathcal{E}$ (not necessarily a topos or a model of type theory), the following.

Definition. *For each $i \in \mathcal{I}$, write $\mathcal{I}^-(i)$ for the full subcategory of $i/\mathcal{I}$ spanned by the strictly degree-lowering maps, write $\partial(i/\mathcal{W}^{-1}\mathcal{I})$ for the full subcategory of $i/\mathcal{W}^{-1}\mathcal{I}$ with the initial object removed, and write $\mathbb{G}_n\mathcal{I}$ for the full subgroupoid of $\mathcal{I}$ spanned by the objects of degree $n \in \mathbb{N}$. Further, for each $i \in \mathcal{I}$, denote by $M_i \colon \mathcal{E}^{\mathcal{I}} \to \mathcal{E}$ the matching object functor.*

Theorem 1. *Assume that, for each $i \in \mathcal{I}$, the restriction of the localisation $\gamma\colon \mathcal{I} \to \mathcal{W}^{-1}\mathcal{I}$ given by $\gamma|_i\colon \mathcal{I}^-(i) \to \partial(i/\mathcal{W}^{-1}\mathcal{I})$ is an initial functor and that all limits indexed by $\mathcal{I}^-(i)$ exist in $\mathcal{E}$. Then, if dependent products along $f\colon B \to A$ in $\mathcal{E}^{\mathcal{W}^{-1}\mathcal{I}}$ exist, one also has dependent products along $\gamma^* f\colon \gamma^* B \to \gamma^* A$ in $\mathcal{E}^{\mathcal{I}}$. Furthermore, the canonical map $\gamma^* \Pi_f \Rightarrow \Pi_{\gamma^* f}\, \gamma^*\colon \mathcal{E}^{\mathcal{W}^{-1}\mathcal{I}}/B \to \mathcal{E}^{\mathcal{I}}/\gamma^* A$ is an isomorphism.*

This result generalises that of [2]; for instance, when $\mathcal{I} = \cdot \to \cdot \to \cdot$, inverting either of the arrows preserves dependent products.

In proving Theorem 1, we show the following.

Lemma 2. *For a functor $F\colon \mathcal{D} \to \mathcal{C}$ that preserves pullbacks, the dependent product in $\mathcal{C} \downarrow F$ along $(f, g)\colon (b, y, \beta) \to (a, x, \alpha)$*

$$\begin{array}{ccc} b & \overset{f}{\dashrightarrow} & a \\ {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle\alpha} \\ Fy & \underset{Fg}{\dashrightarrow} & Fx \end{array}$$

exists provided that dependent products along $f\colon b \to a$ and $Fg\colon Fy \to Fx$ in $\mathcal{C}$, and $g\colon y \to x$ in $\mathcal{D}$ exist.

One concludes Theorem 1 using that Reedy induction in the inverse case, as also noted by [4], allows the construction of indexed diagrams by iterative gluing. Also by way of iterative gluing, we further show the following.

Theorem 3. *Assume all limits indexed by $\mathcal{I}^-(i)$ exist in $\mathcal{E}$ for each $i \in \mathcal{I}$. If $\mathcal{E}$ has a subobject classifier, then so does $\mathcal{E}^{\mathcal{I}}$ provided that either $\mathbb{G}_n\mathcal{I}$ is connected for each $n \in \mathbb{N}$ or $\mathcal{E}$ has an initial object. In addition, and independent of subobject classifiers, dependent products along $f\colon B \to A$ in $\mathcal{E}^{\mathcal{I}}$ exist provided that dependent products along $f_i\colon B_i \to A_i$ and $M_i f\colon M_i B \to M_i A$ in $\mathcal{E}$ exist for each $i \in \mathcal{I}$.*

Our result, in contrast to the type-theoretic results of [2], centres on the logical structure of categories of diagrams; for instance, when $\mathcal{I}$ is an inverse category and $\mathcal{E}$ is a topos then so is $\mathcal{E}^{\mathcal{I}}$.

References

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