Semicartesian categories of relations

B. Lindenhovius

Gejza Jenča[1] (gejza.jenca@stuba.sk)
Slovak University of Technology, Bratislava, Slovakia

Bert Lindenhovius[2] (lindenhovius@mat.savba.sk)
Mathematical Institute of the Slovak Academy of Sciences, Bratislava, Slovakia

Abstract.

Quantization is the process of generalizing mathematical structures to the noncommutative setting. Many quantum phenomena have classical counterparts, and can often be modelled by quantized versions of the mathematical structures modelling these classical counterparts. Recently, several mathematical structures have been quantized via a quantization method based on Weaver's notion of a *quantum relation* between von Neumann algebras [10], which he distilled from his work with Kuperberg on the quantization of metric spaces [9]. Quantum relations can be regarded as noncommutative versions of ordinary relations, and admit a rich relational calculus that allows us to generalize concepts to the noncommutative setting. Building on these concepts, Weaver quantized posets [10] and showed that *quantum graphs* [2], which are used for quantum error correction, can be understood in terms of quantum relations [11].

Von Neumann algebras are rather noncommutative generalizations of measure spaces than of sets. Kornell identified *hereditarily atomic* von Neumann algebras, which are essentially (possibly infinite) sums of matrix algebras, as the proper noncommutative generalizations of sets. For this reason, hereditarily atomic von Neumann algebras are also called *quantum sets*, and the category **qRel** of quantum sets and quantum relations can be regarded as the proper noncommutative generalization of the category **Rel** of sets and binary relations. Unlike the category of all von Neumann algebras and quantum relations, **qRel** is dagger compact closed, just like **Rel**. Together with Kornell and Mislove, the second author investigated the categorical properties of quantum posets in this restricted setting of hereditarily atomic von Neumann algebras [8]. Building on this work, they introduced *quantum cpos*, which are noncommutative versions of ω-complete partial orders (cpos). Ordinary cpos can be used to construct denotational models of ordinary programming languages, and in a similar way, they showed that quantum cpos can be used for the denotational semantics of quantum programming languages [7]. Also building on the definition of quantum posets in the hereditarily atomic setting, we introduced *quantum suplattices* [5], which are noncommutative versions of complete lattices and supremum-preserving maps. For the definition of quantum suplattices, the compact structure of **qRel** seems to be essential.

Categorically, quantization via quantum relations can be understood as the internalization of mathematical structures in the category **qRel**, and many theorems about quantized structures via quantum

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relations rely on the categorical properties of **qRel**. There are several categorical generalizations of the category **Rel** such as allegories [3] or bicategories of relations [1], but unfortunately, **qRel** is not an example of either of them. This is mainly due to the fact that the internal functions of **qRel** form a semicartesian monoidal category rather than a cartesian monoidal category, which reflects the quantum character of **qRel**. Tweaking the definitions of either allegories or bicategories of relations is difficult; their cartesian character seems to be essential.

Therefore, we aim to find a different categorical generalization of **Rel** that would capture **qRel**. We take daggers as a primitive notion, and identify six properties of **qRel** as axioms for our categorical generalization of **Rel**. Similar properties also occur in recent categorical axiomatizations of several dagger categories such as the category **Hilb** and **Rel** [4, 6], and likely will form a subset of the axioms of a future categorical characterization of **qRel**. Hence, we define a *semicartesian category of relations* to be a category **R** such that

(1) **R** is a locally small dagger compact category;
(2) **R** has all small dagger biproducts;
(3) **R** has precisely two scalars;
(4) **R** is a dagger kernel category;
(5) For each object X in **R** there is precisely one morphism $X \to I$ with zero kernel;
(6) For each object X and each projection p on X, $p \geq \mathrm{id}_X$ if and only if $\ker p = 0$.

Here, a *projection* on an object X is a morphism $p : X \to X$ such that $p \circ p = p = p^\dagger$. For the last axiom, we use that the first three axioms imply that **R** is a *quantaloid*, i.e., a category enriched over the category **Sup** of complete lattices and supremum-preserving maps. As another consequence of the axioms, we prove that the homsets of **R** are actually orthomodular lattices. We conclude with a discussion of conditions that assure the existence of a power set construction in semicartesian categories of relations.

References

[1] A. Carboni, R.F.C. Walters, *Cartesian bicategories I*, J. Pure Appl. Alg. 49(1), 11-32, 1987.

[2] R. Duan, S. Severini and A. Winter, *Zero-error communication via quantum channels, noncommutative graphs, and a quantum Lovász number*, IEEE Trans. IT., 59, 1164-1174, 2013.

[3] P. Freyd, A. Scedrov, *Categories, Allegories*, North-Holland, Amsterdam, 1990.

[4] C. Heunen, A. Kornell, *Axioms for the category of Hilbert spaces*, PNAS **119**, e2116285119, 2022.

[5] G. Jenča, B. Lindenhovius, *Quantum Suplattices*, Proceedings of the 20th International Conference on Quantum Physics and Logic, pp. 58–74, 2023.

[6] A. Kornell, *Axioms for the category of sets and relations*, arXiv:2302.14153 [math.CT], 2023.

[7] A. Kornell, B. Lindenhovius, and M. Mislove, *Quantum CPOs*, Proceedings QPL, 2021.

[8] A. Kornell, B. Lindenhovius, and M. Mislove, *A category of quantum posets*, Indagationes Mathematicae **33**(6), 1137–1171, 2022.

[9] G. Kuperberg, N. Weaver, *A von Neumann algebra approach to quantum metrics*, Mem. Amer. Math. Soc. **215**, 2012.

[10] N. Weaver, *Quantum relations*, Mem. Amer. Math. Soc. **215**, 2012.

[11] N. Weaver, *Quantum Graphs as Quantum Relations*, J. Geom. Anal. **31**, 9090–9112, 2021.