Sketches and Classifying Logoi

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Abstract. This talk is based on the preprint [4].

The notion of *sketch* was introduced by Ehrensmann [3, 5]. It consists of a category together with a specification of certain cones and cocones. Using the idea that certain logical operations can be described through limits and colimits, sketches have been considered as one of the many formalisation of the concept of *theory* [2, 7, 8]. In particular, they can be used to present theories in infinitary logics.

The aim of this work is to extend what was done with sites and topoi in the context of geometric logic to infinitary logic introducing the notions of *rounded sketch* and *logos*. More precisely, we want to replicate the pattern that sites are presentations of *geometric* theories and that the classifying topos gives a *syntax independent* avatar of the theory. In a similar way our notion of *rounded sketch* gives the presentation of any infinitary theory (including geometric ones) and the classifying logos its syntax independent presentation.

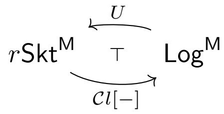
Logic Fragment	**Presentation**	**Morita Classifying Object**
Geometric	Site	Topos
Infinitary	Rounded Sketch	Logos

We start showing some nice (*topological*) properties of the 2-category of sketches, which turn out to be useful for some important constructions. For instance, we give an explicit formula to calculate weighted pseudo co/limits in the 2-category of sketches and we prove that the tensor product for sketches (studied by Benson in [1]) is closed.

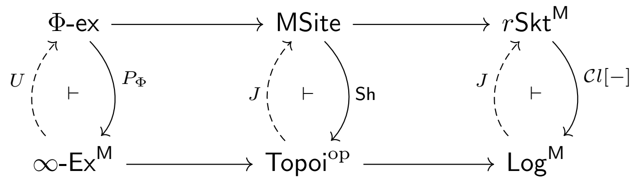
Then, we provide some normalisation constructions which will be useful for our main result, a Diaconescu-like theorem for rounded sketches and logoi. More precisely, for an appropriate notion of *Morita smallness*, we show that for any Morita small sketch $\mathcal{S}$ we can construct its *left sketch classifier* $\hat{\mathcal{S}}$, i.e. a left sketch together with a sketch morphism $J_{\mathcal{S}} \colon \mathcal{S} \to \hat{\mathcal{S}}$ inducing, for any left sketch $\mathcal{M}$, an equivalence as below.

$$- \circ J_{\mathcal{S}} \colon \mathsf{LSkt}(\hat{\mathcal{S}}, \mathcal{M}) \to \mathsf{Skt}(\mathcal{S}, \mathcal{M})$$

Moreover, we use this result to prove that the $\widehat{(-)}$-construction restricted to rounded sketches shows that the 2-category $\mathsf{Log}^{\mathsf{M}}$ of Morita small logoi is (bi)reflective in the 2-category $r\mathsf{Skt}^{\mathsf{M}}$, of Morita small rounded sketches.



This result generalises similar known ones for classifying topoi and Φ-exact categories [6], summarised in the commutative (not considering the dashed arrows) diagram below.



References

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