The Giry monad revisited

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Abstract.

The thesis of this contribution is that one does not have to restrict the class of measurable spaces under consideration to develop a fruitful theory of probability. To this end we suggest a variation of the concept of the Giry monad. As in Giry's legendary paper, we motivate the new concept by study of preservation of limits contributing new results. As an application we show that the Wasserstein distance is actually a distance on the probability measures considered in our set-up.

Motivation.

In 1982 Giry introduced her concept of Giry monad in two variations—first, as a monad on the category <u>Meas</u> of measurable spaces[1] and, second, as a monad on the category of Polish spaces. Both are motivated with preservation of limit properties, which turn out to be stronger in the later case.

Around the same time, some deep exploration of measure theory was still taking place: [1, 2, 3, 5]. Unfortunately, it seems that the relevance of the remarkable and miraculous result of Pachl [2] for a categorical approach to probability was not spotted. Actually, it enables one, to generalise Giry's results for Polish spaces to measurable spaces.

When working with general measurable spaces and measures thereon some limitations occur:

1. projections of measurable sets are not necessarily measurable
2. the Giry monad does not necessarily (weakly) preserve directed limits[2], and
3. countably generated σ-algebras are normally too small to model the notion of "almost surely".

Classically, these issues are addressed by restricting the class of spaces under consideration—with analytic spaces (including Polish spaces) being the most general class that allows for a rich theory (actually, this approach basically transfers limitation 1 into a definition). But analytic spaces do not encompass the theory of distributions, i.e. discrete measures on arbitrary sets, which play a paramount role in logic and computer science. A solution to limitation 3 is to enlarge the σ-algebra by completing it. Though this process comes at a price: Since a countable representation is lost, one is often forced into a situational choice.

Another problem is that quite natural "large" examples are excluded by the classical approach, e.g. the measurable space induced by the well-known French railway metric defined on the set $\mathbb{R}^2$:

$$d(u,v) = \begin{cases} ||u| - |v|| & \text{if } u = rv \text{ for some } r \in (0,\infty) \\ |u| + |v| & \text{else} \end{cases} \quad \text{for } u, v \in \mathbb{R}^2 \tag{1}$$

[1] i.e. pairs $(X, \mathcal{A})$ of a set X and a σ-algebra $\mathcal{A}$

[2] in elementary terms: given consistent probabilities on the objects of a diagram in <u>Meas</u> a probability on the limit need not exist

modelling all potential railway lines in France. As for more categorical limitations, note that analytic spaces can have at most the cardinality of the continuum, so arbitrary limits and colimits are already excluded by size.

We suggest to remedy the situation as follows: Instead of excluding certain measurable spaces, we restrict the Giry monad. Namely, let a **law** on a measurable space $(X, \mathcal{A})$ be a probability measure thereon, such that it extends to a probability measure p' on a larger set $X_p \supseteq X$ on which a σ-algebra $\mathcal{A}_p$ generated from a semicompact paving[3] is given such that $\mathcal{A}$ is a subset of the p'-completion of $\mathcal{A}_p$. This approach generalises the notion of a Radon space. Probably, this idea appeared to one or the other already. But they then failed to form a functor therefrom, as one must guarantee that the push-forward of a law along a measurable map is again a law. Surprisingly, Pachl proved this in 1979 [2] (see also [6, 452R]). So we define the **Giry monad** on $(X, \mathcal{A})$ to be the collection of laws on $(X, \mathcal{A})$.

Results.

As directed limits are not preserved by the Giry monad as defined by Giry [4], she had to impose a technical condition. Only in the case of Polish spaces, which she discusses only for the index set ω^{op}, she could avoid this technical condition. We prove limit preservation in the case of our Giry monad holds for general directed index categories on measurable spaces.

Moreover, we discuss other limit shapes, especially pushouts, where we can provide a result of weak limit preservation. In this context, we can also show that the Wasserstein distance is a distance, i.e. satisfies the triangle inequality, for laws. Classically, the Wasserstein distance is only considered for probability measures on separable metric spaces, where it is a corner stone of several industries—e.g. optimal transport or concurrency theory in computer science. To round up the discussion, we give a negative examples of shapes that are not preserved, e.g. equalisers.

For analytic spaces, our Giry monad coincides with Giry's original definition. The same holds in the example expressed in (1). We also add a few propositions paving the way to a further development of probability (and measure) theory based on the current suggestion.

We conclude with some set theoretic remarks. Moreover, we give some thoughts on how to "extend" our approach to analytic spaces, i.e. define a measure that looks like a measure on an analytic space.

The speaker is supported by EPSRC NIA Grant EP/X019373/1

References

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[3] also called *countably compact* or *semicompact class*, a collection of subsets of X such that every countable subcollection satisfying the finite intersection property (*FIP*, i.e. every finite subcollection has non-empty intersection) has non-empty intersection