$\mathscr{V}$-graded categories and $\mathscr{V}$-$\mathscr{W}$-bigraded categories: Functor categories and bifunctors over non-symmetric bases

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Abstract. Categories graded by a monoidal category $\mathscr{V}$, or ($\mathscr{V}$-)*graded categories*, were introduced by Wood [1] under the name *large $\mathscr{V}$-categories*, and they simultaneously generalize both $\mathscr{V}$-enriched categories and $\mathscr{V}$-actegories in the absence of any assumptions on $\mathscr{V}$; also see [2, 3, 4]. Explicitly, $\mathscr{V}$-graded categories may be defined as categories enriched in the presheaf category $\hat{\mathscr{V}} = [\mathscr{V}^{\mathsf{op}}, \mathrm{SET}]$ with its Day convolution monoidal structure, while they also admit a direct elementwise definition.

Given an arbitrary strict monoidal category $\mathscr{V}$, we show that $\mathscr{V}$-graded categories support a robust theory of graded functor categories and bifunctors, enabled by a notion of *bigraded category* that we introduce. This is in contrast with the usual settings of enriched category theory, where the definition of enriched functor categories and bifunctors employs a symmetry [5], braiding, or duoidal structure on $\mathscr{V}$ [6] and so is not applicable when working with just a biclosed monoidal category $\mathscr{V}$ or, more generally, a closed bicategory, where one nevertheless has a robust theory of $\mathscr{V}$-modules and of $\mathscr{V}$-categories of $\mathscr{V}$-valued presheaves [7, 8], but these are not defined in terms of bifunctors and functor categories.

We develop our results on graded functor categories in a general setting that begins with a given pair of strict monoidal categories $\mathscr{V}$ and $\mathscr{W}$. Writing $\mathscr{W}^{\mathsf{rev}}$ to denote the reverse of $\mathscr{W}$, we consider both $\mathscr{V}$-graded categories and $\mathscr{W}^{\mathsf{rev}}$-graded categories, calling the former *left $\mathscr{V}$-graded categories* and the latter *right $\mathscr{W}$-graded categories.* A *$\mathscr{V}$-$\mathscr{W}$-bigraded category* is then a left $(\mathscr{V} \times \mathscr{W}^{\mathsf{rev}})$-graded category and so has both an underlying left $\mathscr{V}$-graded category and an underlying right $\mathscr{W}$-graded category. For example, both $\mathscr{V}$ and $\hat{\mathscr{V}}$ underlie $\mathscr{V}$-$\mathscr{V}$-bigraded categories.

Given a left $\mathscr{V}$-graded category $\mathscr{A}$ and a $\mathscr{V}$-$\mathscr{W}$-bigraded category $\mathscr{C}$, we show that there is a right $\mathscr{W}$-graded category $[\mathscr{A}, \mathscr{C}] = {}^{\mathscr{V}}[\mathscr{A}, \mathscr{C}]_{\mathscr{W}}$ whose objects are (left) $\mathscr{V}$-graded functors from $\mathscr{A}$ to $\mathscr{C}$. Similarly, given a right $\mathscr{W}$-graded category $\mathscr{B}$ and a $\mathscr{V}$-$\mathscr{W}$-bigraded category $\mathscr{C}$, we obtain a left $\mathscr{V}$-graded category $[\mathscr{B}, \mathscr{C}] = {}_{\mathscr{V}}[\mathscr{B}, \mathscr{C}]^{\mathscr{W}}$ whose objects are right $\mathscr{W}$-graded functors from $\mathscr{B}$ to $\mathscr{C}$. In particular, if $\mathscr{D}$ is a left $\mathscr{V}$-graded category, then its *opposite* $\mathscr{D}^{\mathsf{op}}$ is a *right* $\mathscr{V}$-graded category, so if $\mathscr{C}$ is a $\mathscr{V}$-$\mathscr{V}$-bigraded category then $[\mathscr{D}^{\mathsf{op}}, \mathscr{C}]$ is a left $\mathscr{V}$-graded category.

Given a left $\mathscr{V}$-graded category $\mathscr{A}$ and a right $\mathscr{W}$-graded category $\mathscr{B}$, we construct a $\mathscr{V}$-$\mathscr{W}$-bigraded category $\mathscr{A} \boxtimes \mathscr{B}$ whose objects are pairs (A, B) with $A \in \mathsf{ob}\,\mathscr{A}$ and $B \in \mathsf{ob}\,\mathscr{B}$. Given also a $\mathscr{V}$-$\mathscr{W}$-bigraded category $\mathscr{C}$, we may therefore consider $\mathscr{V}$-$\mathscr{W}$-bigraded functors of the form $F : \mathscr{A} \boxtimes \mathscr{B} \to \mathscr{C}$, which provide a notion of *bifunctor* in the graded setting. Writing ${}_{\mathscr{V}}\mathrm{GCAT}$, $\mathrm{GCAT}_{\mathscr{W}}$, and ${}_{\mathscr{V}}\mathrm{GCAT}_{\mathscr{W}}$ for the 2-categories of left $\mathscr{V}$-graded categories, right $\mathscr{W}$-graded categories, and $\mathscr{V}$-$\mathscr{W}$-bigraded categories, respectively, we show that there are 2-natural isomorphisms

$${}_{\mathscr{V}}\mathrm{GCAT}(\mathscr{A}, [\mathscr{B}, \mathscr{C}]) \cong {}_{\mathscr{V}}\mathrm{GCAT}_{\mathscr{W}}(\mathscr{A} \boxtimes \mathscr{B}, \mathscr{C}) \cong \mathrm{GCAT}_{\mathscr{W}}(\mathscr{B}, [\mathscr{A}, \mathscr{C}]) .$$

In the special case where $\mathscr{V}$ is *symmetric* monoidal and we take $\mathscr{W} = \mathscr{V}$, there is no essential distinction between left and right $\mathscr{V}$-graded categories, while every $\mathscr{V}$-graded category is canonically $\mathscr{V}$-$\mathscr{V}$-bigraded, and we recover the $\mathscr{V}$-graded functor categories and bifunctors that were studied by Wood [1, §1.6] and coincide with the usual $\hat{\mathscr{V}}$-enriched concepts for the symmetric monoidal category $\hat{\mathscr{V}} = [\mathscr{V}^{\mathsf{op}}, \mathrm{SET}]$, though $\mathscr{A} \boxtimes \mathscr{B}$ does not coincide with the monoidal product of $\hat{\mathscr{V}}$-categories $\mathscr{A} \otimes \mathscr{B}$.

Given an arbitrary strict monoidal category $\mathscr{V}$ and a pair of *right* $\mathscr{V}$-graded categories $\mathscr{A}$ and $\mathscr{B}$, we may consider $\mathscr{V}$-$\mathscr{V}$-bigraded functors $F : \mathscr{B}^{\mathsf{op}} \boxtimes \mathscr{A} \to \mathscr{C}$ valued in any $\mathscr{V}$-$\mathscr{V}$-bigraded category $\mathscr{C}$, and we call these *$\mathscr{V}$-graded modules* from $\mathscr{A}$ to $\mathscr{B}$ in $\mathscr{C}$. Passing to the special case where $\mathscr{C} = \hat{\mathscr{V}}$, we show that $\mathscr{V}$-graded modules in $\hat{\mathscr{V}}$ are precisely $\hat{\mathscr{V}}$-modules between $\hat{\mathscr{V}}$-categories, in the sense obtained by specializing [7, 8] to base of enrichment $\hat{\mathscr{V}} = [\mathscr{V}^{\mathsf{op}}, \mathrm{SET}]$. Furthermore, we show that the $\hat{\mathscr{V}}$-enriched presheaf $\hat{\mathscr{V}}$-category $\mathcal{P}\mathscr{B}$ that is obtained by applying Street's enriched presheaf construction [7] relative to the base of enrichment $\hat{\mathscr{V}}$ is precisely the right $\mathscr{V}$-graded category $[\mathscr{B}^{\mathsf{op}}, \hat{\mathscr{V}}]$ that is obtained as an example of the above general construction of graded functor categories.

References

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