On the representability of actions of non-associative algebras

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Abstract.
It is well known that in the semi-abelian category **Grp** of groups, internal actions are represented by automorphisms. This means that the category **Grp** is *action representable* and the representing object, which is called the *actor*, is the group of automorphisms. Another example of action representable category is the variety **Lie** of Lie algebras over a fixed field $\mathbb{F}$, with the actor of a Lie algebra $\mathfrak{g}$ being the Lie algebra of derivations $\mathrm{Der}(\mathfrak{g})$. The notion of action representable category has proven to be quite restrictive: for instance, if a non-abelian variety $\mathcal{V}$ of non-associative algebras over an infinite field $\mathbb{F}$, with $\mathrm{char}(\mathbb{F}) \neq 2$, is action representable, then $\mathcal{V} = \mathbf{Lie}$. More recently G. Janelidze introduced the notion of *weakly action representable category*, which includes a wider class of categories, such as the variety **Assoc** of associative algebras and the variety **Leib** of Leibniz algebras.

In this talk we show that for an *algebraically coherent* and *operadic* variety $\mathcal{V}$ and an object X of $\mathcal{V}$, it is always possible to construct a *partial algebra* $\mathcal{E}(X)$, called *external weak actor* of X, and a natural monomorphism of functors

$$\tau\colon \mathrm{Act}(-, X) \rightarrowtail \mathrm{Hom}_{\mathbf{PAlg}}(U(-), \mathcal{E}(X)),$$

where **PAlg** is the category of partial algebras over $\mathbb{F}$ and $U\colon \mathcal{V} \to \mathbf{PAlg}$ denotes the forgetful functor. The pair $(\mathcal{E}(X), \tau)$ is called *external weak representation* of the functor $\mathrm{Act}(-, X)$. Moreover, for any other object B of $\mathcal{V}$, we provide a complete description of the morphisms $(B \to \mathcal{E}(X)) \in \mathrm{Im}(\tau_B)$, i.e. of the homomorphisms of partial algebras which identify the actions of B on X in $\mathcal{V}$, and we show that the existence of a weak representation is closely connected to the *amalgamation property*, which we use to prove that the variety **CAssoc** of commutative associative algebras is weakly action representable.

Eventually, we give an application of the construction of the external weak actor in the context of varieties of *unital* algebras, which are *ideally exact categories* in the sense of G. Janelidze: we prove that, if $\mathcal{V} = \mathbf{Alt}$ is the variety of *alternative* algebras and X is a unital alternative algebra, then $\mathcal{E}(X) \cong X$ is the actor of X. In other words, unital alternative algebras, such as the algebra $\mathbb{O}$ of *octonions*, have representable actions.

This is joint work with J. Brox, (*Universidad de Valladolid*, Spain), Alan S. Cigoli (*Università degli Studi di Torino*, Italy), Xabier García Martínez (*Universidade de Vigo*, Spain), Giuseppe Metere (*Università degli Studi di Milano*, Italy), Tim Van der Linden and Corentin Vienne (*Université catholique de Louvain*, Belgium).

References

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