From Yoneda's additive regular spans to fibred cartesian monoidal opfibrations: a route towards a 2-dimensional cohomology of groups

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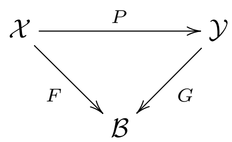
Abstract.

In his pioneering 1960 paper [7], Nobuo Yoneda introduced the notion of additive regular span in developing a formal categorical setting useful to classify n-fold extensions in a suitable additive context. As an application of his Additive Classification Theorem, he proved that functors $\mathbf{Ext}^n : \mathcal{A}^{op} \times \mathcal{A} \to \mathcal{A}b$, defined through connected components of n-fold extensions, are additive in both variables, with $\mathcal{A}$ abelian category. This result made it possible to introduce cohomology groups in abelian categories without projectives or injectives.

When moving to a non-additive context, as the category **Gp** of groups, to interpret cocycles we need *crossed* n-fold extensions (this approach was adopted also to give an interpretation of cohomology groups in an intrinsic context, as the one of strongly semi-abelian categories in [6], see also [1]). This change carries out the necessity of breaking the symmetry in Yoneda's setting and the need of giving a fibrational interpretation of regular spans, as decribed in [2].

The main goal of this talk is to show how it is possibile to extend Yoneda's *Additive* Classification Theorem in two different directions. Indeed, we are able to provide an abstract setting which from one side allows to treat the non-additive case, such as the one in **Gp**. On the other, it makes it possible to grow up in dimension, obtaining symmetric 2-groups in place of just abelian groups.

The key notion we eventually need is that of *fibred cartesian monoidal opfibration*



which turns out to be a cartesian object in the 2-category $\mathsf{OpFib}(\mathsf{Fib}(\mathcal{B}))$ of internal opfibrations in the category of fibrations over a fixed category $\mathcal{B}$. After providing a characterization of such morphisms $P\colon (\mathcal{X}, F) \to (\mathcal{Y}, G)$ in $\mathsf{OpFib}(\mathsf{Fib}(\mathcal{B}))$, it turns out that, for each b in $\mathcal{B}$, the restriction

$$P_b\colon \mathcal{X}_b \to \mathcal{Y}_b$$

is a cartesian monoidal opfibration, as defined in [4].

Joint work with A. S. Cigoli and G. Metere

In particular, when for each b in $\mathcal{B}$, P_b is an opfibration fibred in groupoids and $\mathcal{Y}_b$ is an additive category, then it is possible to show that each fibre of P_b is endowed with a structure of symmetric 2-group and, as b varies in $\mathcal{B}$, "change-of-base" functors are symmetric monoidal.

Now, suppose we start with the morphism in $\mathsf{Fib}(\mathbf{Gp})$

$$\begin{array}{ccc} \mathbf{XExt} & \xrightarrow{\ \Pi\ } & \mathbf{Mod} \\ & {\scriptstyle \Pi_0}\searrow \quad \swarrow{\scriptstyle (\_)_0} & \\ & \mathbf{Gp} & \end{array}$$

where **XExt** is the category of crossed extensions of groups

$$X\colon \qquad 0 \longrightarrow A \xrightarrow{j} G_2 \xrightarrow{\partial} G_1 \xrightarrow{p} B \longrightarrow 1 \tag{1}$$

Mod is the category of group-modules, and the functors are given by: $\Pi(X) = (B, A)$, $\Pi_0(X) = B$, $(B, A)_0 = B$.

Here we are not in the situation described above, since, for any group B the restrictions Π_B are not fibred in groupoids. But, we can move in the desired context by factorizing Π through a suitable category of fractions, as proved in [3]. This way we can apply the results above and define, for any B-module A, its symmetric 2-group of cohomology $\mathbb{H}^3(B, A)$, whose structure is described in [5].

References

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