Non-cartesian internalisation and enriched quasi-categories

A. Mertens

Arne Mertens (arne.mertens@uantwerpen.be)
University of Antwerp

Abstract.

The classical nerve functor $N : \mathrm{Cat} \hookrightarrow \mathrm{SSet}$ characterises categories as simplicial sets satisfying a unique lifting condition. Relaxing the uniqueness leads to the definition of a quasi-category as a simplicial set satisfying the weak Kan condition. If one wants to consider enriched quasi-categories, a natural question to ask is the following: Given a suitable monoidal category $(\mathcal{V}, \otimes, I)$, what is the nerve of a $\mathcal{V}$-enriched category? One might expect this to result in a simplicial object in $\mathcal{V}$, but a quick observation will show that no reasonable functor landing in simplicial objects $\mathcal{V}\,\mathrm{Cat} \to S\mathcal{V}$ exists when $\mathcal{V}$ is not cartesian monoidal (e.g. when $\mathcal{V}$ the category of vector spaces with the tensor product).

In this talk, I will present *tensor-simplicial* or *templicial objects* $S_{\otimes}\mathcal{V}$ as a generalisation of simplicial sets in the non-cartesian context. This is joint work with Wendy Lowen [11], and it fits the following scheme:

$$\begin{array}{ccccc}
\text{Categories internal to } \mathcal{V} & \underset{\text{if } \otimes = \times}{\simeq} & \text{Categories internal to } (\mathcal{V}, \otimes, I) & \supset & \mathcal{V}\text{-enriched categories} \\
{\scriptstyle N_{\mathcal{V}}}\big\downarrow & & {\scriptstyle N_{\mathcal{V}}}\big\downarrow & & {\scriptstyle N_{\mathcal{V}}}\big\downarrow \\
\text{Simplicial objects in } \mathcal{V} & \underset{\text{if } \otimes = \times}{\simeq} & \text{Colax monoidal functors } \boldsymbol{\Delta}_f^{op} \to \mathcal{V} & \supset & \text{Templicial objects in } \mathcal{V}
\end{array}$$

where each $N_{\mathcal{V}}$ is a fully faithful right-adjoint generalising N. Categories internal to a (not necessarily cartesian) monoidal category $(\mathcal{V}, \otimes, I)$ were introduced by Aguiar in [1]. They recover classical internal categories when $\mathcal{V}$ is cartesian monoidal, and contain $\mathcal{V}$-enriched categories as a subclass. This picture extends nicely to higher dimensions. Let $\boldsymbol{\Delta}_f \subseteq \boldsymbol{\Delta}$ be the monoidal category of finite intervals, then a colax monoidal functor $\boldsymbol{\Delta}_f^{op} \to \mathcal{V}$ may be considered as a "simplicial object internal to $(\mathcal{V}, \otimes, I)$". It was already observed by Leinster [10] that these precisely recover $S\mathcal{V}$ when $\mathcal{V}$ is cartesian.

We posit templicial objects as a suitable context to define an enriched variant of Joyal's quasi-categories [8]. In particular this is motivated by (noncommutative) algebraic geometry - where the strictly enriched model of dg-categories play a central role - and algebraic deformation theory [3]. Our main results are the following, some of which I will outline during the talk:

1. In [11], we identified an analogue of the weak Kan condition for templicial objects. If it is satisfied we call the templicial object a *quasi-category in* $\mathcal{V}$, which precisely recovers classical quasi-categories when $\mathcal{V} = \mathrm{Set}$. To express this condition, we make essential use of the necklaces of [2][6]. In a separate project with Violeta Borges Marques [4], we construct a Reedy structure on necklaces.

2. Employing necklaces, we construct a general framework for producing enriched variants of other nerves as well, such as the homotopy-coherent nerve [5], the dg-nerve [13] and the cubical nerve

[9], all of which land in $S_{\otimes}\mathcal{V}$. This moreover allows to obtain explicit descriptions of their left-adjoints.

3. Let k be a commutative ring. Through the enriched variant of the dg-nerve, we show in [12] an equivalence of categories between non-negatively graded dg-categories over k and quasi-categories in k-modules equipped with a certain Frobenius structure.

4. Our current main goal is to construct a model structure on the category of templicial objects - analogous to Joyal's model structure on simplicial sets - such that the enriched homotopy coherent-nerve $N^{hc}_{\mathcal{V}} : S\mathcal{V}\text{-Cat} \to S_{\otimes}\mathcal{V}$ becomes a Quillen equivalence. This would establish quasi-categories in $\mathcal{V}$ as a model for $S\mathcal{V}$-enriched ∞-categories in the sense of [7]. Moreover, we expect templicial objects to be model monoidal, which fails for $S\mathcal{V}$-Cat.

   I will outline some results in this direction. For instance, when $\mathcal{V} = \text{Vect}(k)$, templicial vector spaces define a category of cofibrant objects with a compatible symmetric monoidal structure.

References

[1] M. Aguiar, *Internal categories and quantum groups*, ProQuest LLC, Ann Arbor, M, 1997.

[2] H.J. Baues, *Geometry of loop spaces and the cobar construction*, Mem. Amer. Math. Soc., 25 (1980), ix+171.

[3] V. Borges Marques, W. Lowen and A. Mertens, Deformations of quasi-categories in modules, preprint arXiv:2308.09148, 2023.

[4] V. Borges Marques and A. Mertens, *The category of necklaces is Reedy monoidal*, Theory Appl. Categ., 41 (2024), no. 3, 71–85.

[5] J.-M. Cordier, *Sur la notion de diagramme homotopiquement cohérent*, Cahiers de Topologie et Géom. Différentielle Catégoriques, 23 (1982), no. 1, 93–112.

[6] D. Dugger and D Spivak *Rigidification of quasi-categories*, Algebr. Geom. Topol., 11 (2011), no. 1, 225–261.

[7] D. Gepner and R. Haugseng, *Enriched ∞-categories via non-symmetric∞-operads*, *Adv. Math.*, 279 (2015), 575–716.

[8] A. Joyal, The theory of quasi-categories and its applications. *unpublished*, 2008.

[9] B. Le Grignou, *Cubical model categories and quasi-categories*, Theory Appl. Categ., 35 (2020), no. 18, 622–664.

[10] T. Leinster, Homotopy algebras for operads, preprint arXiv:math/0002180v1, 2000.

[11] W. Lowen and A. Mertens, *Enriched quasi-categories and the templicial homotopy coherent nerve*, Alg. Geom. Topol., accepted for publication, 2024.

[12] W. Lowen and A. Mertens, Frobenius templicial modules and the dg-nerve, preprint, arXiv:2005.04778v3, 2023.

[13] Lurie, J. Higher algebra. *unpublished*, 2017.