Aspects of 2-dimensional Elementary Topos Theory

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Abstract.

We will present the main results of our PhD thesis, that contribute to expand 2-dimensional elementary topos theory. As elementary topos theory has had an enormous success, with numerous applications to geometry and logic, we believe it is very fruitful to generalize the theory to dimension 2. Introduced by Weber in [5], 2-dimensional elementary topos theory is still at its beginning, but with a high potential of application to stacks, classifying topoi and 2-categorical logic. Lawvere's idea of an elementary topos was that of a generalized universe of sets. On this line, an elementary 2-topos is a generalized universe of categories.

We will focus in particular on the concept of 2-classifier, which was introduced by Weber in [5] and is the 2-categorical generalization of the notion of subobject classifier. The idea is to classify discrete opfibrations, that have higher dimensional fibres. And the archetypal example is given by the category of elements construction, exhibiting $\mathcal{C}at$ as the archetypal elementary 2-topos. So, interestingly, a 2-classifier can also be thought of as a Grothendieck construction inside a 2-category. We will introduce a notion of good 2-classifier, that captures well-behaved 2-classifiers and is closer to the point of view of logic. The idea is to still have as classifier an object of generalized truth values together with the choice of a verum, as in dimension 1. This is realized by upgrading the classification process from one regulated by pullbacks (in dimension 1) to one regulated by comma objects (in dimension 2).

We will present a novel technique of reduction of the study of 2-classifiers to dense generators, developed in our [4]. Dense generators capture the idea of a family of objects that generate all the other ones via nice colimits; the preeminent example is given by representables in categories of presheaves. We will show that both the conditions of 2-classifier and what gets classified by a 2-classifier can be checked just over the objects that form a dense generator. This substantially reduces the work needed to prove that something is a 2-classifier. For example, applied to the archetypal case of $\mathcal{C}at$, it allows us to deduce all the major properties of the category of elements construction from the trivial observation that everything works well over the singleton category.

We will then apply the theorems of reduction of 2-classifiers to dense generators to produce a good 2-classifier in stacks, classifying all discrete opfibrations with small fibres. This generalizes to dimension 2 the fundamental result that Grothendieck topoi are elementary topoi. Indeed, Grothendieck topoi are given by categories of sheaves, and stacks are precisely the 2-categorical generalization of sheaves. Stacks still capture the idea to glue together compatible local data into a global datum, but the compatibility conditions are only required up to isomorphism. They are a key object of study of the modern algebraic geometry, and they have solved numerous problems (e.g. moduli problems) that were not solvable using ordinary spaces or 1-dimensional sheaves. Thanks to our results, the 2-categories of stacks, i.e. Grothendieck 2-topoi, will be elementary 2-topoi.

To reach our good 2-classifier in stacks, we will first apply our theorems of reduction of 2-classifiers to dense generators to produce a good 2-classifier in prestacks (i.e. 2-presheaves). We will achieve this by using an indexed version of the Grothendieck construction, developed in our joint work with Caviglia [1]. This gives a pseudonatural equivalence of categories between opfibrations over a fixed base in the 2-category of 2-copresheaves and 2-copresheaves on the Grothendieck construction of the fixed base. Our result can be interpreted as the result that every (op)fibrational slice of a Grothendieck 2-topos is a Grothendieck 2-topos. So this generalizes to dimension 2 the so-called fundamental theorem of elementary topos theory, in the Grothendieck topoi case. Our good 2-classifier in prestacks involves a 2-dimensional generalization of the concept of sieve, that is a key element of the notion of Grothendieck topology. We will then restrict our good 2-classifier in prestacks to one in stacks, via factorization arguments and our theorems of reduction to dense generators. In particular, we will generalize closedness of a sieve to dimension 2. Our results also solve a problem posed by Hofmann and Streicher in [2] when attempting to lift Grothendieck universes to sheaves.

The driving idea behind our technique of reduction to dense generators is to express an arbitrary object as a nice colimit of the dense generators and induce the required data using the universal property of the colimit. In order to handle such colimits in our 2-categorical setting, we apply the calculus of colimits in 2-dimensional slices that we developed in [3]. In particular, our calculus generalizes to dimension 2 the well-known fact that a colimit in a 1-dimensional slice category is precisely the map from the colimit of the domains of the diagram that is induced by the universal property. We show that the appropriate 2-dimensional slice to consider for this is the lax slice, and that the appropriate 2-dimensional colimits to consider are marked conical colimits.

References

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