Toposes as Lex-Presentable Categories

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Abstract.

When learning topos theory, one encounters a bewildering array of *exactness conditions* — special interactions between (weighted) colimits and finite limits which obtain in the category of sets, but not in general categories. These are all of generally the same form, but they differ enough in the details to challenge the memory. Categories with some of these exactness conditions (regular, coherent, geometric, exact, lextensive, etc.̃) play an important role in topos theory, and these sorts of categories correspond to doctrines of geometric logic or type theory.

In their beautiful paper "Lex colimits", Garner and Lack smooth this story out by showing that these exactness conditions emerge by co-completing not in the 2-category **Cat** of categories, but instead relative to KZ-doctrines on the 2-category **Lex** of categories with finite limits and finite limit preserving functors between them. They are therefore able to reframe Giraud's characterization of sheaf toposes in the following way: a topos is a lex-cocomplete category which is (locally) presentable. However, while the notion of lex-cocompletion takes place in **Lex**, the notion of presentability is the original notion which takes place in **Cat**. Can we give a characterization of toposes as lex-presentable categories taking place fully in **Lex**?

In their delightful paper "Accessibility and presentability in 2-categories", Di Liberti and Loregian define presentability relative to a *KZ-context* $\nu : \mathcal{S} \hookrightarrow \mathcal{P}$ (a fully faithful inclusion of KZ-doctrines) on an arbitrary 2-category. When $\mathcal{S}$ has a complementary KZ-context $\mathcal{D} \hookrightarrow \mathcal{P}$ so that the induced map $\mathcal{S}\mathcal{D} \to \mathcal{P}$ is an equivalence (and under a few other minor assumptions), Di Liberti and Loregian provide a Gabriel-Ulmer duality between *petite* $\mathcal{D}$-cocomplete objects and $\mathcal{S}$-presentable objects in this abstract setting.

In this talk, we will apply Di Liberti and Loregian's theory to the 2-category **Lex** and the KZ-doctrine $\mathcal{P}$ of free cocompletion, interpreting the results using Garner and Lack's theory of lex colimits to see lex-κ-presentable categories as κ-coherent toposes. We'll begin by defining a notion of weighted colimit relative to a KZ-doctrine, which will bring Di Liberti and Loregian's abstract setting closer to Garner and Lack's formulation of lex cocompletion using weighted colimits. This will also allow us to describe taking lex colimits "one-by-one" and not just the operation of lex-cocompletion under a class. We will then discuss the conditions under which two classes of weights $\mathcal{S}$ and $\mathcal{D}$ (to equivocate between the classes and their free cocompletion KZ-doctrines, for the moment) are *complementary* and satisfy the assumptions necessary to apply Di Liberti and Loregian's Gabriel-Ulmer duality to produce a duality between $\mathcal{D}$-theories and $\mathcal{S}$-presentable toposes.

In the proof of their Theorem 6.4, Garner and Lack define a sub-canonical topology $j_{\mathcal{D}}$ on $\mathcal{D}$-cocomplete categories C for which the Yoneda embedding $y : C \to \mathbf{Sh}(C, j_{\mathcal{D}})$ is $\mathcal{D}$-cocontinuous. We will say that $\mathcal{S}$ and $\mathcal{D}$ are *complementary classes of weights* when (1) any colimit may be expressed

as an $\mathcal{S}$-weighted colimit of $\mathcal{D}$-weighted colimits and (2) a $\mathcal{S}$-weighted colimit (in presheaves) of $j_{\mathcal{D}}$-sheaves is still a $j_{\mathcal{D}}$-sheaf. We'll see that complementary classes of colimits satisfy Di Liberti and Loregian's Assumptions 3.14 and therefore admit a Gabriel-Ulmer duality.

We'll conclude by observing that κ-filtered colimits ($\mathcal{S}$) are complementary to κ-small colimits ($\mathcal{D}$), so that Di Liberti-Loregian's Gabriel-Ulmer duality gives us the familiar dual equivalence between κ-exact categories (κ-complete pretoposes)[1] with lex κ-cocontinuous functors between them and κ-coherent toposes with relatively κ-tidy geometric morphisms between them.[2][3]

References

[1] R. Garner, and S. Lack, *Lex colimits*, Journal of Pure and Applied Algebra 216 (2012), no. 6, 1372–1396.

[2] I. Di Liberti, and F. Loregian *Accessibility and presentability in 2-categories*, Journal of Pure and Applied Algebra 227 (2023), no. 1.

[3] I. Di Liberti; Ramos González, Julia. Gabriel-Ulmer duality for topoi and its relation with site presentations. Appl. Categ. Structures 28 (2020), no. 6, 935–962.

[1] This is true for $\kappa \geq \omega$; for $\kappa = 1$, $\mathcal{D}$ is the identity and $\mathcal{S} = \mathcal{P}$ is free cocompletion and the duality is between small lex categories and free toposes.

[2] I believe that this duality between pretoposes and coherent toposes was first observed by Makkai, but I could not find a reference.

[3] We note that this Gabriel-Ulmer duality differs slightly from that of [3] which begins with ordinary Gabriel-Ulmer duality and carves it down to toposes. They establish a duality with *proto-toposes* rather than pretoposes.