Double categories of relations relative to factorization systems

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Abstract.

Relations and spans in a category have been studied in the context of double categories. The recent work [1] investigated a characterization of the double category of spans in a finitely complete category, and later in [4], the double category of relations in a regular category was characterized. In this talk, we will present a generalization of these results by introducing *double categories of relations relative to stable orthogonal factorization systems*.

Characterization of double categories of relations. Our presentation will be based on the notion of relations relative to a factorization system (E, M) on a category $\mathbf{C}$, or M-relations, which are defined as spans jointly belonging to the right class M. Taking arrows as tight arrows (vertical arrows) and M-relations as loose arrows (horizontal arrows), we obtain the double category of M-relations. For a double category $\mathbb{D}$ with a terminal object, we can define a class $\mathrm{Fib}(\mathbb{D})$ of tight arrows called *fibrations*, which in the case of the double category of profunctors, are precisely the discrete fibrations. We will explain the conditions on a double category $\mathbb{D}$ under which the class of fibrations $\mathrm{Fib}(\mathbb{D})$ becomes the right class of a stable orthogonal factorization system (E, M) on the category of tight arrows, and the double category $\mathbb{D}$ is equivalent to the double category of M-relations.

Layering classes of factorization systems via double categorical properties. We will also discuss some important properties of factorization systems that are reflected in the double categorical viewpoint. Our characterization theorem translates properties of factorization systems into properties of double categories, and vice versa, as shown in Table 1. The property of *unit-pureness* on double categories will be explored, which is the counterpart of the property of a factorization system having the left class included in the class of epimorphisms. By the correspondence, we reprove the characterization of the double category of spans and that of relations in a regular category from the unified viewpoint, providing a rationale that some conditions, such as unit-pureness and local posetality, are essential in the original proofs.

Cauchy condition on the double category of M-relations. Another property of double categories we will discuss is the Cauchy condition. A category is Cauchy complete if any left adjoint profunctor from (or into, depending on the convention) it is representable. This leads to the definition of Cauchy double categories ([5]) that all loose adjoints in it are representable by tight arrows. We will show that in a unit-pure double category of M-relations, left adjoint loose arrows are "ana-tight-arrows", meaning that they are of the form

$$A \xleftarrow{m} B \xrightarrow{f} C$$

where m is a monomorphism belonging to the left class E of the factorization system. It brings us to the characterization of unit-pure Cauchy double categories of relations as those whose right class M of the corresponding factorization system includes all monomorphisms. As previously noted in [6], the Cauchy condition is the categorical formulation of the unique-choice principle.

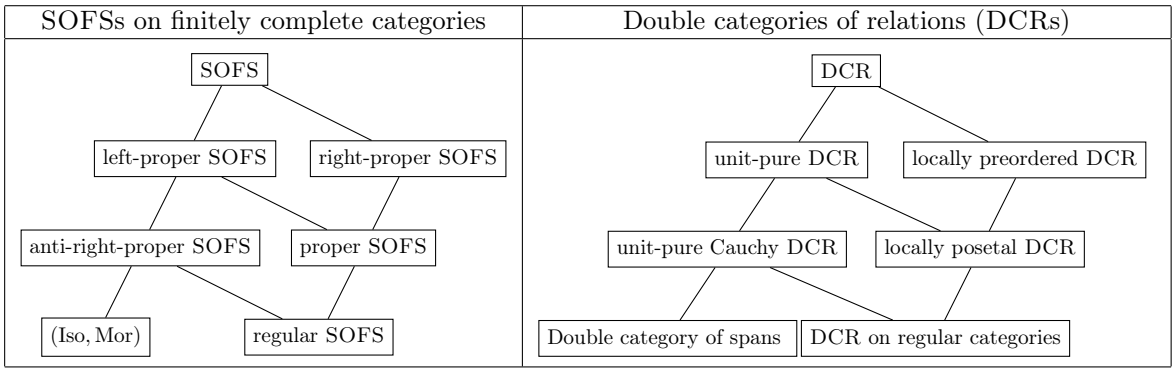

Table 1: Correspondence between classes of stable orthogonal factorization systems and double categories of relations.

References

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