Coalgebraic enrichment of categorical W-types

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Abstract.

In both the traditions of functional programming and categorical logic, one takes the perspective that most data types should be obtained as initial algebras of polynomial endofunctors. For instance, the natural numbers are obtained as the initial algebra of the endofunctor $X \mapsto X + 1$, assuming that the category in question (often the category of sets) has a terminal object 1 and a coproduct $+$. Much theory has been developed around this approach, which culminates in the notion of W-types [2, 3].

In another tradition, that of categorical algebra, algebras (in the traditional sense) over a field k are studied. It has been long understood (going back at least to Wraith and Sweedler, according to [1]) that the category of k-algebras is naturally enriched over the category of k-coalgebras, a fact which has admitted generalization to several other settings (e.g. [1, 5]). Here, we generalize those classic results to the setting of an endofunctor on a category, and in particular those endofunctors that are considered in the theory of W-types.

That is to say, this work is the beginning of a development of an analogue of the theory of W-types – not based on the notion of initial objects in a *category* of algebras, but rather on a generalized notion of initial object in a *coalgebra enriched category* of algebras. The hom-coalgebras of our enriched category carry more information than the hom-sets in the unenriched category that is usually considered in the theory of W-types. We are then able to generalize the notion of *initial algebra*, taking inspiration from the theory of weighted limits, which is more expressive, and thus can be used to specify more objects than the usual notion of initial algebra. Because of our move to the enriched setting, then, we have better control than in the unenriched setting, and we are able to specify more data types than just those which are captured by the theory of W-types.

Our main theorem is the following.

Theorem. *Let* $(\mathsf{C}, \otimes, \mathbb{I}, \underline{\mathsf{C}}(-,-))$ *be a locally presentable symmetric monoidal closed category. Let* $F : \mathsf{C} \to \mathsf{C}$ *be an accessible lax symmetric monoidal endofunctor. Then the category* Alg_F *of* F*-algebras is enriched, tensored, and powered over the symmetric monoidal category* CoAlg_F *of* F*-coalgebras.*

We show that many endofunctors of interest in the theory of W-types satisfy these hypotheses. For instance, Set is a locally presentable symmetric monoidal closed category. The following functors

on a locally presentable symmetric monoidal closed category satisfy the hypotheses: the identity functor, any constant functor at a commutative monoid, the coproduct of two functors that satisfy the hypothesis, and the product two functors that satisfy the hypotheses.

In particular, the functor $X \mapsto X + 1$ on Set satisfies the hypotheses, and we work out very explicitly what the enrichment (and tensoring and cotensoring) tells in this situation. In this concrete case, we see that the enrichment encodes a notion of *partial algebra homomorphism*, whereas the usual category of algebras encodes the notion of *total algebra homomorphism*.

We then observe that there is an implicit parameter in the notion of initial algebra which we may now vary. One might think of an initial object as a certain *colimit*, but in reality, an initial object in a category C is usually (equivalently) defined as an object I with the property that $\mathsf{hom}(I, X) = \{*\}$ for every $X \in \mathsf{C}$. That is, I is the vertex of a cone over the identity functor on C with the special property that each leg of the cone (at an object $X \in \mathsf{C}$) is the only morphism of $\mathsf{hom}(I, X)$. The reader might know that as such, an initial object can always be defined as the *limit* of the identity functor on C. Now that we are in the enriched setting, however, the appropriate notion of limit becomes that of *weighted limit* in which we are able ask not just that $\mathsf{hom}(I, X) = \{*\}$ but that $\mathsf{hom}(I, X) = W$ for any object W. Thus, we make the following definition.

Definition. *Consider a monoidal category* $(\mathsf{C}, \otimes, \mathbb{I}, \underline{\mathsf{C}}(-,-))$ *and endofunctor* $F : \mathsf{C} \to \mathsf{C}$ *satisfying the hypotheses of the above theorem.*

For $W \in \mathsf{CoAlg}_F$*, we define the* W*-initial algebra to be the limit of the identity functor on* Alg_F *(viewed as the enriched categories described in the above theorem) weighted by the constant functor* $\mathsf{Alg}_F \to \mathsf{CoAlg}_F$ *at* W*.*

With this, we are able to obtain algebras that represent, for instance in the example of the functor $X \mapsto X + 1$, partial induction.

Our hope is that all of this extra structure discovered in the quite classical categorical semantics of functional programming languages can be used to augment them.

This talk reports on [4] and further work in progress.

References

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