On a (terminally connected, pro-etale) factorization system for geometric morphisms

A. Osmond

Olivia Caramello (`Olivia.caramello@igrothendieck.org`)
Istituto Grothendieck

Axel Osmond (`axelosmond@orange.fr`)
Istituto Grothendieck

Abstract.

A classical result of topos theory states that any locally connected geometric morphism $f : \mathcal{F} \to \mathcal{E}$ factorizes uniquely as a connected geometric morphism followed by an etale geometric morphism, where the etale part is presented by the essential image $f_!(1_{\mathcal{F}})$ of the terminal object – morally, the object of connected components of the image of f. Such a factorization still makes sense for the wider class of essential geometric morphisms, though the left part of this factorization is no longer connected; however it still is *terminally connected*.

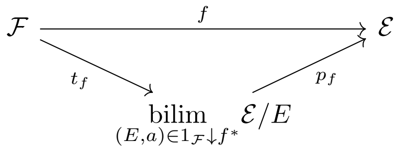
Terminally connected geometric morphisms are in some sense those which are only connected "from the point of view of the terminal object". In the essential world, this condition (as it was first introduced in [1]) says that the essential image preserves the terminal object; but a reformulation of this condition, which makes sense for arbitrary geometric morphisms, is that there exists a natural isomorphism $\mathcal{F}[1_{\mathcal{F}}, f^*(-)] \simeq \mathcal{E}[1_{\mathcal{E}}, -]$, or in words, that the inverse image uniquely lifts global elements. Terminally connected geometric morphisms have some interesting stability properties along Beck-Chevalley squares or also left to bicomma squares (though not along bipullback); moreover one can show that they are exactly those that are left orthogonal to etale geometric morphisms, which suggests they form the left class of a factorization system for all geometric morphisms.

This requires first to identify a correct generalization of etale morphisms on the right. Indeed, in full generality a geometric morphism may lack an essential image part displaying connected components; yet its inverse image part, as a lex functor, nevertheless possesses a *left pro-adjoint*, that is, a relative left adjoint along its free completion under cofiltered limits. In some sense, though the connected components of an arbitrary geometric morphism may not be indexed by a discrete set internal to the codomain topos, they will nevertheless form a pro-discrete internal locale. This suggests that a correct replacement of etale morphisms could be provided by those that are cofiltered limits of etale geometric morphisms, a.k.a *pro-etale geometric morphisms.*

Factorizing a geometric morphism $f : \mathcal{F} \to \mathcal{E}$ through the etale geometric morphism $\mathcal{E}/E \to \mathcal{E}$ at a given object E of $\mathcal{E}$ amounts to providing a global element $a : 1_{\mathcal{F}} \to f^*E$; similarly factorizations through a pro-etale geometric morphism correspond to cofiltered diagrams in the category of elements $1_{\mathcal{F}} \downarrow f^*$. In particular there is a best such factorization through the pro-etale indexed by the cofiltered category of all global elements of the inverse image part; though this category is large, this cofiltered limit is always well-defined for $1_{\mathcal{F}} \downarrow f^*$ can be shown to admits a small initial subcategory thanks to

an accessibility argument. In particular any pro-etale geometric morphism can be reindexed by the category of elements of its own inverse image, which provides a canonical presentation.

If now one factorizes a geometric morphism through the pro-etale morphism indexed by all the possible global elements of the inverse image



the residual left part t_f is then always terminally connected, for in some sense all global elements are displayed in a faithful way in the cofiltered limit, a fact that can be proven concretely thanks to a formula presenting the bilimit topos as the sheaf topos over the pseudocolimits of etale sites. This begets an orthogonal *(terminally connected, pro-etale)* factorization for all geometric morphisms. A peculiar manifestation of this factorization was already known since [2] as the *Grothendieck-Verdier localization* at a point of a topos, the germ at this point.

A remarkable property of this factorization is that it allows for a canonical factorizations of lax 2-cells. In general 2-dimensional factorization systems do not come with a canonical way to relate the factorizations of two 1-cells related by a 2-cell; equivalently, the orthogonality condition for (pseudo)squares does not extend to either lax or oplax squares. But in this very case, it happens to be so: terminally connected morphisms enjoy a special lax orthogonality condition relative to pro-etale, which comes also with special cocomma stability property and factorization of 2-cells. This property is somewhat reminiscent of the comprehensive factorization (*initial, discrete opfibrations*) on **Cat** and we will see this is not a coincidence.

This talk will describe properties of terminally connected and pro-etale geometric morphisms, and give a throughout proof of the existence of their associated factorization. We will also give a more intrinsic characterization of pro-etale geometric morphisms as those whose inverse image generates the domain topos through fibers of global elements. We will also give a closely related factorization for locales morphisms and discuss their behavior along the localic reflection. We also discuss some syntax-semantics aspects and the relation with another factorization, the (*focalisation*, *terminally connected*) in **Lex** that is implicitly involved in Gabriel-Ulmer duality. We finally discuss the relation with another possible generalization of the (connected, etale) factorization, the *(connected, algebraic)* factorization identified in [3].

References

[1] O. Caramello, *Denseness conditions, morphisms and equivalences of toposes*, preprint arXiv:1906.08737, 2020.

[2] P.T. Johnstone, I. Moerdijk, *Local maps of toposes*, Proceedings of the London Mathematical Society, vol. s3-58, 1989.

[3] J. Lurie, *Higher topos theory*, Princeton University Press, 2009.